Random Matrix, Singularities and Open/Close Intersection Numbers

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Abstract

The s-point correlation function of a Gaussian Hermitian random matrix theory, with an external source tuned to generate a multi-critical singularity, provides the intersection numbers of the moduli space for the p-th spin curves through a duality identity. For one marked point, the intersection numbers are expressed to all order in the genus by Bessel functions. The matrix models for the Lie algebras of $O(N)$ and $Sp(N)$ provide the intersection numbers of non-orientable surfaces. The Kontsevich-Penner model, and higher p-th Airy matrix model with a logarithmic potential, are investigated for the open intersection numbers, which describe the topological invariants of non-orientable surfaces with boundaries. String equations for open/closed Riemann surface are derived from the structure of the s-point correlation functions. The Gromov-Witten invariants of CP^1 model are evaluated for one marked point as an application of the present method.

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1 Introduction

The intersection numbers of moduli space of Riemann surface are topological invariants, which are closely related to universal singularities in random matrix models with an external source. By tuning appropriately the matrix source, multi-critical behaviors are obtained at the edge of the density of state [1, 2, 3]. They are described by Airy and higher Airy kernels for the p-th degenerate singularity. The correlation functions for the p-th singularity turn out to be generating functions for the intersection numbers of p-spin curves on Riemann surfaces [4, 5, 6, 7]. In this article, we extend our previous work on the calculation of the intersection numbers of p spin curves. This technique relies on a duality [4, 6] which is specified in the next section.

Recently, the open intersection theory with boundaries has been investigated in [16, 17, 18]. The generating matrix model for these open intersection numbers is the Kontsevich-Penner model [21, 22], namely Kontsevich's model with a logarithmic potential, studied before in [19, 20]. The Virasoro equations for this case provide a different structure from KdV, since the Riemann surface with boundaries is no longer orientable. The expansion of the logarithmic matrix model, like a ribbon graph expansion of Kontsevich matrix model, provides the non-orientable surface. The situation is similar to that of non-orientable surfaces generated by the matrix models of $O(N)$ and $Sp(N)$ Lie algebras [12]. We compute here the open intersection numbers, and higher p-th spin curves, including such Lie algebras.

The composition of this article is the following. In section 2, several dualities exchanging the size of the random matrices in an external matrix with the number of points in correlation functions, are recalled. The definition of the intersection numbers is briefly recalled. In section 3, the intersection numbers for $p = 2, 3, 4, 5$, and $p = -1$ (Euler characteristics) and one marked point, are explicitly given in terms of Bessel functions. In section 4, the intersection numbers for the Lie algebras of O(2N), O(2N+1), and Sp(N) are discussed. The Euler characteristics for those non-orientable surfaces follow. In section 5, the open intersection numbers are computed for one marked point from the Kontsevich-Penner model , and the relation to the $O(N)$ model for non-orientable intersection numbers is discussed. In section 6, the intersection numbers for multiple marked points are evaluated and the relation to Virasoro equations is examined. Section 7 is application of the present method to the Gromov-Witten invariants of CP^1. Section 8 is devoted to discussions. In an appendix we study the case of p-spin curves for open Riemann surfaces.

2 Dualities

- **GUE ensemble**

We have discussed in earlier publications the possibility of computing topological invariants relative to Riemann surfaces using a Gaussian ensemble of $N \times N$ Hermitian random matrices with appropriately tuned external matrix sources A. The method relies on two basic ingredients : i) a totally explicit formula for the K-point correlation functions for arbitrary given source matrix A, based on the HarishChandra-Itzykson-Zuber integral over the unitary group[1], ii) a duality for the correlation functions of K characteristic polynomials $< \prod_1^K \det(\lambda_\alpha - M) >_A$, with a probability distribution

$$P_A(M) = \frac{1}{Z_N} \epsilon^{-\frac{1}{2}\mathrm{trM}^2 - \mathrm{trMA}} \tag{2.1}$$

This duality exchanges the size N of the matrices with K, the number of points, i.e. the $N \times N$ Hermitian random matrices are replaced by $K \times K$ Gaussian random matrices; the $N \times N$ source matrix A is exchanged with the $K \times K$ source matrix Λ whose eigenvalues are the λ_α : it reads [4]

$$\begin{aligned} &\frac{1}{Z_N} \int d^{N^2} M \prod_{\alpha=1}^{K} \det(\lambda_\alpha - M) \epsilon^{-\frac{1}{2}\mathrm{tr(M+A)}^2} \\ &= (-i)^{NK} \frac{1}{Z_K} \int d^{K^2} B \prod_{i=1}^{N} \det(a_i \delta_{\alpha\beta} - B_{\alpha\beta}) \epsilon^{-\frac{1}{2}\mathrm{tr(B+i\Lambda)}^2} \end{aligned} \tag{2.2}$$

This duality is clearly well adapted to the large N limit since the r.h.s. is an integral over matrices whose size is independent of N. But we want to briefly summarize how else we have used it.

Tuning appropriately the eigenvalues of the source matrix A, one can obtain in the dual version the Airy matrix model which was introduced by Kontsevich [8] ; it appears here as an edge singularity, reminiscent of the Tracy-Widom kernel[9] which governs the vicinity of the edge of Wigner's semi-circle. This is done by taking for the source matrix A the identity matrix and considering the large N scaling regime in which the λ_k are close to one, namely $N^{2/3}(\lambda_k - 1)$ finite, the r.h.s. of (2.2) becomes the Airy matrix integral introduced by Kontsevich. In this regime one finds

$$\frac{1}{Z_N} \int d^{N^2} M \prod_{\alpha=1}^{K} \det(\lambda_\alpha - M) \epsilon^{-\frac{1}{2}\mathrm{tr(M+A)}^2}$$

$$= e^{\frac{N}{2}\mathrm{tr}\Lambda^2} \int d^{K^2} B e^{i\frac{N}{3}\mathrm{trB^3 + iNtrB(\Lambda-1)}} \tag{2.3}$$

and the r.h.s. after a rescaling $B \to BN^{-1/3}$, $(\lambda - 1) \to (\lambda - 1)N^{-2/3}$ reduces to Kontsevich Airy integral ;

$$Z_{KT} = \int dB e^{\frac{i}{3}\mathrm{trB^3 - tr\Lambda B^2}} \tag{2.4}$$

In the r.h.s. of (2.3) tuning the a_α one may generate the Airy matrix integral, whereas the l.h.s. is still a Gaussian integral whose correlation functions are known explicitly. Indeed the one-point function with the probability weight (2.1) is given by [4]

$$U(\sigma) = \langle \mathrm{tr} e^{\sigma \mathrm{M}} \rangle = \frac{1}{\sigma} \oint \frac{\mathrm{du}}{2\mathrm{i}\pi} \mathrm{e}^{\sigma \mathrm{u}} \prod_1^{\mathrm{N}} \frac{\mathrm{u} - \mathrm{a}_\alpha + \sigma}{\mathrm{u} - \mathrm{a}_\alpha} \tag{2.5}$$

in which the a_α are the eigenvalues of the source matrix A. This formula is exact for any N. For the s-point function [4] the result is an integral over s complex variables

$$\begin{aligned} U(\sigma_1 \cdots \sigma_s) &= \langle \mathrm{tr} e^{\sigma_1 \mathrm{M}} \cdots \mathrm{tr} e^{\sigma_s \mathrm{M}} \rangle \\ &= e^{\frac{1}{2}\sum_1^s \sigma_i^2} \oint \prod_1^s \frac{du_i e^{\sigma_i u_i}}{2i\pi} \det \frac{1}{u_i + \sigma_i - u_j} \prod_{i=1}^s \prod_{\alpha=1}^N (1 + \frac{\sigma_i}{u_i - a_\alpha}) \end{aligned} \tag{2.6}$$

Let us illustrate on the simplest example how one can use (2.3) and (2.6), for the one-point function. We take $\lambda_a = \lambda$ for $a = 1 \cdots K$. Then we are dealing with

$$< [\det(\lambda - M)]^K >_A = < [\det(1 - iB)]^N >_\Lambda \tag{2.7}$$

and make use of "replicas", i.e. of the identity

$$\lim_{K \to 0} \frac{1}{K} \frac{d}{d\lambda} [\det(\lambda - M)]^K = \mathrm{tr} \frac{1}{\lambda - \mathrm{M}} \tag{2.8}$$

Since λ is in the vicinity of the edge of Wigner's semi-circle, the resolvent has to be computed in this regime, but knowing explicitly $U(\sigma)$, this is straightforward.

The intersection numbers of moduli space of curves are defined as coefficients in the expansion of $t_n = \mathrm{tr}\frac{1}{\Lambda^{n+\frac{1}{2}}}$ for Z_{KT}. The coefficients , the intersection numbers $< \tau_{n_1} \cdots \tau_{n_s} >_g$, are defined also as

$$< \tau_{n_1} \cdots \tau_{n_s} >_g = \int_{\overline{M}_{g,s}} \psi_1^{n_1} \cdots \psi_s^{n_s} \tag{2.9}$$

where ψ_i is called as ψ class and equal to $c_1(\mathcal{L}_i)$ with c_1 first Chern class and $\mathcal{L}_i$ is line bundle at marked point i. We have shown before that $U(\sigma_1,...\sigma_s) =< \mathrm{tr}e^{\sigma_1 M}\cdots \mathrm{tr}e^{\sigma_s \mathrm{M}} >$ is generating function of the intersection numbers, since it is a Fourier transform of the density correlation functions,

$$U(\sigma_1,...\sigma_s) =< \int\prod_i^s d\lambda_i e^{\sigma_i\lambda_i}\mathrm{tr}\delta(\lambda_\mathrm{i}-\mathrm{M}) >_\mathrm{A} \tag{2.10}$$

This function provides a polynomial expansion of σ_i. The degree of total σ_i is equal to $\sum_i^s(n_i+\frac{1}{2})$, and $\sum_i^s n_i = 3g-3+s$. This is similar to the evaluation of the intersection numbers by hyperbolic surfaces [25], by replacing the parked points by disks, whose perimeter lengths are $l_1,....,l_s$ and the generating function of the intersection numbers is a polynomial of l_i, (i=1,...,s) and the total degree is $6g-6+2s$ [26].

We have shown earlier, using this strategy together with replicas, how to compute from there the intersection numbers of the moduli of curves on Riemann surfaces with one marked point [4] ; the method clearly allowed for more marked points. Higher multi-critical singularities, characterized by an integer p may also be tuned from appropriately chosen external sources A [2, 3]. One may obtain thereby a generalized p-th Airy matrix model [6], taking $A = \mathrm{diag}(a_1,...,a_1,....,a_{p-1},....,a_{p-1})$, with $(p-1)$ distinct eigenvalues values, each of them being $(\frac{N}{p-1})$ times degenerate. The conditions[2] are

$$\sum_{\alpha=1}^{p-1}\frac{1}{a_\alpha^2} = p-1,\ \ \sum_{\alpha=1}^{p-1}\frac{1}{a_\alpha^m} = 0\ \ (m=3,...,p),\ \ \sum_{\alpha=1}^{p-1}\frac{1}{a_\alpha^{p+1}} \neq 0 \tag{2.11}$$

and we obtain the p-th degenerated Airy matrix model. The correlation functions are known in integral form

$$\begin{aligned}&U(\sigma_1,...,\sigma_n)\\ &= \oint\prod\frac{du_i}{2\pi}e^{-\frac{N}{p^2-1}\sum_{\alpha=1}^{p-1}\frac{1}{a_\alpha^{p+1}}[(\sum_{i=1}^n(u_i+\frac{1}{2N}\sigma_i)^{p+1}-\sum_{i=1}^n(u_i-\frac{1}{2N}\sigma_i)^{p+1})]}\\ &\times\prod\det(\frac{1}{u_i-u_j+\sigma_i})\end{aligned} \tag{2.12}$$

The one point function, which corresponds to one marked point, becomes in the scaling limit

$$U(\sigma) = \frac{1}{N\sigma}\int\frac{du}{2\pi i}e^{-\frac{c}{p+1}[(u+\frac{1}{2}\sigma)^{p+1}-(u-\frac{1}{2}\sigma)^{p+1}]} \tag{2.13}$$

with $c = \frac{N}{p-1}\sum_{\alpha=1}^{p-1}\frac{1}{a_\alpha^{p+1}}$. The Airy matrix model corresponds to the $p=2$ case. It is also possible to continue the model to negative values of

p. In particular the case $p=-1$ provides a generating function for the orbifold Euler characteristics of surfaces with n marked points, allowing us to recover through this method the classic results of [10, 11, 6].

- **Lie algebras of classical groups**

  The previous duality (2.2) extends to Lie algebras of the classical groups, such as antisymmetric real matrices for the orthogonal group $O(2N)$.

$$< \prod_{\alpha=1}^{k} \det(\lambda_\alpha \cdot \mathrm{I} - X) >_A = < \prod_{n=1}^{N} \det(a_n \cdot \mathrm{I} - Y) >_\Lambda \tag{2.14}$$

where X is $2N \times 2N$ real antisymmetric matrix ($X^t = -X$) and Y is $2k \times 2k$ real antisymmetric matrix ; the eigenvalues of X and Y are thus pure imaginary. A is also a $2N \times 2N$ antisymmetric matrix, and it couples to X as an external matrix source. The matrix Λ is $2k \times 2k$ antisymmetric matrix, coupled to Y. We assume, without loss of generality, that A and Λ have the canonical form :

$$A = \begin{pmatrix} 0 & a_1 & 0 & 0 & \cdots \\ -a_1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & a_2 & 0 \\ 0 & 0 & -a_2 & 0 & 0 \\ \cdots & & & & \end{pmatrix}, \tag{2.15}$$

i.e.

$$A = a_1 v \oplus \cdots \oplus a_N v, \qquad v = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.16}$$

Λ is expressed also as

$$\Lambda = \lambda_1 v \oplus \cdots \oplus \lambda_k v. \tag{2.17}$$

The characteristic polynomial $\det(\lambda \cdot \mathrm{I} - X)$ has the $2N$ roots, $(\pm i\lambda_1, \cdots, \pm i\lambda_n)$. The Gaussian averages in (4.1) are defined as

$$\begin{aligned} < \cdots >_A &= \frac{1}{Z_A} \int dX e^{\frac{1}{2}\mathrm{tr} X^2 + \mathrm{trXA}} \\ < \cdots >_\Lambda &= \frac{1}{Z_\Lambda} \int dY e^{\frac{1}{2}\mathrm{tr} Y^2 + \mathrm{trY\Lambda}} \end{aligned} \tag{2.18}$$

in which X is a $2N \times 2N$ real antisymmetric matrix, and Y a $2k \times 2k$ real antisymmetric matrix ; the coefficients Z_A and Z_Λ are such that the expectation values of one is equal to one. The derivation relies on a representation of the characteristic polynomials in terms of integrals

over Grassmann variables, as for the $U(N)$ case, but it is more involved [12].

Here again the Harish Chandra formula leads to explicit formulae for the correlation functions. The one-point function for instance is

$$U(s) = -\frac{1}{Ns}\oint \frac{dv}{2\pi i}\prod_{n=1}^{N}\left(\frac{v^2+a_n^2}{(v+\frac{s}{2})^2+a_n^2}\right)\frac{v+\frac{s}{2}}{v+\frac{s}{4}}e^{vs+\frac{s^2}{4}}, \tag{2.19}$$

and higher point functions are also known explicitly. Therefore one may repeat the same tuning plus duality strategy in this case, leading to the desired topological numbers for non-orientable surfaces generated by these antisymmetric matrix models.

- **Superduality**

Consider

$$F_{P,Q}(\lambda_\alpha \cdots \mu_\beta \cdots) = \frac{1}{Z_N} < \frac{\prod_{\alpha=1}^{P}\det(\lambda_\alpha - M)}{\prod_{\alpha=1}^{Q}\det(\mu_\beta - M)} >_A \tag{2.20}$$

with

$$\langle \mathcal{O}(M)\rangle_A = \frac{1}{Z_A}\int dM\mathcal{O}(M)e^{-\frac{1}{2}\mathrm{trM^2+trMA}} \tag{2.21}$$

For instance the average resolvent is given by $P = Q = 1$, after taking derivative with respect to λ and setting $\lambda = \mu$.

Let us recall standard definitions for supermatrices: let

$$X = \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} \tag{2.22}$$

in which the matrix elements of a and b are commuting numbers, those of α and β anticommuting. Then the supertrace

$$\mathrm{str}X = \mathrm{tr}a - \mathrm{tr}b \tag{2.23}$$

ensures the cyclic invariance. The superdeterminant is given by

$$\mathrm{sdet}X = \frac{\det a}{\det(b-\beta a^{-1}\alpha)} = \frac{\det(a-\alpha b^{-1}\beta)}{\det b} \tag{2.24}$$

based on the integral

$$\int d\theta d\bar{\theta}dxd\bar{x}\epsilon^{i\overline{\Phi}X\Phi} = (\mathrm{sdet}X)^{-1} \tag{2.25}$$

where

$$\Phi = \begin{pmatrix} x \\ \theta \end{pmatrix} \qquad \overline{\Phi} = \begin{pmatrix} \bar{x} & \bar{\theta} \end{pmatrix} \tag{2.26}$$

The formulae are obtained either by integrating first the commuting variables, or the anticommuting variables first. We use the conventions

$$\overline{\theta_1 \theta_2} = \bar{\theta}_1 \bar{\theta}_2 \tag{2.27}$$

and

$$\bar{\bar{\theta}} = \theta \tag{2.28}$$

Finally the usual bosonic formula still holds here, namely

$$\mathrm{str}(\log X) = \log(\mathrm{sdetX}). \tag{2.29}$$

We are now in position to derive the duality formula for (2.30) which we first write in integral form as

$$F_{P,Q}(\lambda_\alpha \cdots \mu_\beta \cdots) = \int \prod_{a=1}^{N} \prod_{\alpha=1}^{Q} \prod_{\beta=1}^{P} d\bar{x}_\alpha^a dx_\alpha^a d\bar{\theta}_\beta^a d\theta_\beta^a \\ \langle e^{-\sum_{\alpha=1}^{P} \bar{x}_\alpha^a (\lambda_\alpha - M)_{ab} x_\alpha^b - \sum_{\beta=1}^{Q} \bar{\theta}_\beta^a (\mu_\beta - M)_{ab} \theta_\beta^b} \rangle_A \tag{2.30}$$

or, introducing the $(Q+P) \times (Q+P)$ diagonal matrix Λ made of μ_β , $\beta = 1 \cdots Q$ and λ_α , $\alpha = 1 \cdots P$

$$F_{P,Q}(\lambda_\alpha \cdots \mu_\beta \cdots) = \int \prod_{a=1}^{N} d\bar{x}_\alpha^a dx_\alpha^a d\bar{\theta}_\beta^a d\theta_\beta^a \langle e^{-\ \bar{\Phi}^a \Lambda \Phi^a + \bar{\Phi}^a \cdot M_{ab} \Phi^b} \rangle_A \tag{2.31}$$

Since

$$\langle e^{\mathrm{trXM}} \rangle_A = e^{\frac{1}{2}\mathrm{trX}^2 + \mathrm{trAX}} \tag{2.32}$$

we have

$$X_{ba} = \bar{\Phi}^a \cdot \Phi^b = \sum_{\alpha=1}^{P+Q} \bar{\Phi}_\alpha^a \Phi_\alpha^b. \tag{2.33}$$

Then

$$\mathrm{tr(AX)} = \sum_{\mathrm{n}=1}^{\mathrm{N}} \mathrm{a_n} \sum_{\alpha=1}^{\mathrm{P+Q}} \bar{\Phi}_\alpha^{\mathrm{n}} \Phi_\alpha^{\mathrm{n}} \tag{2.34}$$

in which the a_n are the eigenvalues of A,

$$\mathrm{trX}^2 = \sum_{\mathrm{a,b}=1}^{\mathrm{N}} \sum_{\alpha,\beta=1}^{\mathrm{P+Q}} \bar{\Phi}_\alpha^{\mathrm{a}} \Phi_\alpha^{\mathrm{b}} \bar{\Phi}_\beta^{\mathrm{b}} \Phi_\beta^{\mathrm{a}}. \tag{2.35}$$

Let us define the matrix Γ, $(Q+P)\times(Q+P)$

$$\Gamma_{\alpha,\beta} = \sum_{a=1}^{N} \overline{\Phi^a_\alpha}\Phi^a_\beta = \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ \Gamma_2^\dagger & \Gamma_3 \end{pmatrix} = \begin{pmatrix} \bar{x}\cdot x & \bar{\theta}\cdot x \\ \bar{x}\cdot\theta & \bar{\theta}\cdot\theta \end{pmatrix} \tag{2.36}$$

This matrix Γ_1 is Hermitian but Γ_3 is anti-Hermitian.

To express trX^2 in terms of the matrix Γ some commutations are required and one obtains easily

$$\mathrm{trX}^2 = \sum_{\alpha,\beta} = \mathrm{str}(\Gamma^2) \tag{2.37}$$

Therefore

$$\begin{aligned} F_{P,Q}(\lambda_\alpha\cdots\mu_\beta\cdots) \qquad &= \int\prod_{a=1}^{N} d\bar{x}^a_\alpha dx^a_\alpha d\bar{\theta}^a_\beta d\theta^a_\beta \\ & e^{-\ \bar{\Phi}^a\Lambda\Phi^a + \sum_{n=1}^{N} a_n \sum_{\alpha=1}^{P+Q} \bar{\Phi}^n_\alpha\Phi^n_\alpha + \frac{1}{2}\mathrm{str}\Gamma^2} \end{aligned} \tag{2.38}$$

The SUSY Hubbard-Stratonovich transformation reads

$$\int d\Delta e^{\mathrm{str}(-\frac{1}{2}\Delta^2+\Delta\Gamma)} = e^{\frac{1}{2}\mathrm{str}\Gamma^2} \tag{2.39}$$

in which Δ is $(P+Q)\times(P+Q)$ and like Γ as far as hermiticity is concerned. Then

$$\begin{aligned} F_{P,Q}(\lambda_\alpha\cdots\mu_\beta\cdots) = \int d\Delta\int\prod_{a=1}^{N} d\bar{x}^a_\alpha dx^a_\alpha d\bar{\theta}^a_\beta d\theta^a_\beta \\ e^{-\ \bar{\Phi}^a\Lambda\Phi^a + \sum_{n=1}^{N} a_n \sum_{\alpha=1}^{P+Q} \bar{\Phi}^n_\alpha\Phi^n_\alpha} e^{-\frac{1}{2}\mathrm{str}\Delta^2 + \mathrm{str}\Delta\Gamma} \end{aligned} \tag{2.40}$$

One can integrate out on the x's and θ's. The quadratic form in the exponential is

$$-\bar{\Phi}^a\Lambda\Phi^a + \sum_{n=1}^{N} a_n \sum_{\alpha=1}^{P+Q} \bar{\Phi}^n_\alpha\Phi^n_\alpha + \Delta_{\alpha,\beta}\bar{\Phi}^a_\beta\Phi^a_\alpha(-1)^{F_\beta}$$

in which $F_\beta = 0$ for $1\le\beta\le P$ or $F_\beta = 1$ for $(P+1)\le\beta\le(P+Q)$ The integration then gives

$$\prod_{1}^{N} s\det\ {}^{-1}[(\Lambda_\alpha - a_n)\delta_{\alpha\beta} - \Delta_{\alpha\beta}(-1)^{F_\beta}]$$

Therefore we change $\Delta_{\alpha\beta}(-1)^{F_\beta}\to\tilde{\Delta}_{\alpha\beta}$ and one verifies that

$$s\mathrm{tr}\Delta^2 = \mathrm{str}\tilde{\Delta}^2 \tag{2.41}$$

. Then one ends up with

$$F_{P,Q}(\lambda_\alpha \cdots \mu_\beta \cdots) = \int d\Delta e^{-\frac{1}{2}\mathrm{str}\Delta^2} \prod_1^N s\det\ ^{-1}[(\Lambda_\alpha - a_n)\delta_{\alpha\beta} - \Delta_{\alpha\beta}] \tag{2.42}$$

The above identity (2.42) relates an ordinary integral to a super matrix integration. In this sense it is not a full duality although it can be used for the large N-limit or for a super-generalization of the Kontsevich model. However a full superduality has been derived by Desrosiers and Eynard for expectation values of ratios of super-determinants [13] and our identity appears as a simple limiting case.

- **Arbitrary** β
  An extension of the GUE duality (2.2) to the three classical Gaussian ensembles GOE, GUE, GSE with respectively $\beta = 1, 2, 4$ has been derived by Desrosiers [14] , but it exchanges β to $4/\beta$. However the lack of HarishChandra formula for integrating over the orthogonal or symplectic group does not allow one to compute explicitly the k-point functions and we cannot repeat the steps that we have followed for $\beta = 2$. However we have used supergroup methods to obtain the one and two-point functions [5, 15, 44].

3 GUE

$X \in U(N)$

We now use the duality formula for computing the one-point function $U(\sigma)$ for some symmetric spaces.
(i) Let us first consider the Hermitian case:
From the duality formula (2.20), we obtain

$$\begin{aligned} F_N(\lambda,\mu) &= < \frac{\det(\lambda - X)}{\det(\mu - X)} >_A \\ &= \delta_{\lambda,\mu} + N(\lambda-\mu)\frac{e^{-N(\mu^2-\lambda^2)}}{2\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dtdu \prod_{\alpha=1}^N (\frac{a_\alpha - it}{a_\alpha + u})\frac{1}{u-it} \\ &\times\ e^{-\frac{N}{2}u^2 - \frac{N}{2}t^2 - iNt\lambda + Nu\mu} \end{aligned} \tag{3.1}$$

The density of state $\rho(\lambda)$ is [15]

$$\rho(\lambda) = -\lim_{\mu\to\lambda}\frac{1}{\pi N}\frac{\partial}{\partial\mu}\mathrm{Im}F_N(\lambda,\mu)$$

$$= \frac{1}{N}\int \frac{dt}{2\pi i}\oint \frac{du}{2\pi i}\prod_{\alpha=1}^{N}(\frac{a_\alpha - it}{a_\alpha + u})\frac{1}{u-it}e^{-\frac{N}{2}u^2-\frac{N}{2}t^2-iNt\lambda+Nu\lambda} \tag{3.2}$$

By tuning the external source a_α as (2.11), and taking Fourier transform of $\rho(\lambda)$,

$$U(\sigma) = \frac{1}{N\sigma}\oint \frac{du}{2\pi i}e^{-\frac{c}{p+1}[(u+\frac{1}{2}\sigma)^{p+1}-(u-\frac{1}{2}\sigma)^{p+1}]} \tag{3.3}$$

which is identical to the previous expression (2.13).

(1) p=2

The integral (3.3) becomes Gaussian (c=1),

$$\begin{aligned} U(\sigma) &= \frac{1}{\sigma}e^{-\frac{\sigma^3}{12}}\int_{-\infty}^{\infty}\frac{du}{2\pi}e^{-\sigma u^2} \\ &= \frac{1}{2\pi\sigma}\sqrt{\frac{\pi}{\sigma}}e^{-\frac{1}{12}\sigma^3} \end{aligned} \tag{3.4}$$

This may be expressed as a modified Bessel function $K_{\frac{1}{2}}(z)$

$$K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}e^{-z} \tag{3.5}$$

and we have

$$U(\sigma) = \frac{1}{2\pi\sqrt{6}}K_{\frac{1}{2}}(\frac{\sigma^3}{12}) \tag{3.6}$$

This explicit representation gives the intersection numbers fro Riemann surfaces of genus g,

$$< \tau_m >_g = \frac{1}{(24)^g g!}, \quad (m = 3g-2) \tag{3.7}$$

(2) p=3

Then

$$U(\sigma) = \frac{1}{\sigma}\oint \frac{du}{2\pi i}e^{-\sigma u^3 - \frac{1}{4}\sigma^3 u} \tag{3.8}$$

or changing u to $u = v^{1/3}$,

$$U(\sigma) = \frac{1}{3\sigma}\oint \frac{dv}{2\pi i}v^{-\frac{2}{3}}e^{-\sigma v - \frac{1}{4}\sigma^3 v^{\frac{1}{3}}} \tag{3.9}$$

The path integral may be divided into two integrals on the real axis, above and below the cut :

$$\begin{aligned} U(\sigma) &= U_I(\sigma) + U_{II}(\sigma) \\ U_I(\sigma) &= \frac{1}{3\sigma i}\int_0^\infty \frac{dv}{2\pi}(e^{2\pi i}v)^{-\frac{2}{3}}e^{-\sigma v - \frac{1}{4}\sigma^3(e^{2\pi i}v)^{\frac{1}{3}}} \\ U_{II}(\sigma) &= -\frac{1}{3\sigma i}\int_0^\infty \frac{dv}{\pi}(e^{-2\pi i}v)^{-\frac{2}{3}}e^{-\sigma v - \frac{1}{4}\sigma^3(e^{-2\pi i}v)^{\frac{1}{3}}} \end{aligned} \tag{3.10}$$

U_{II} is the complex conjugate of U_I, and $U(\sigma)$ is real. The integer powers of σ, i.e. σ^n, cancel. This corresponds to the spin $j=2$, since $\sigma^{n+\frac{1+j}{3}} = \sigma^{n+1}$. This cancellation means that there is no Ramond term in $U(\sigma)$, and only Neveu-Schwarz types exist.

We have for $p=3$

$$\begin{aligned} U(\sigma) &= (\frac{\sin\frac{\pi}{3}}{\pi})[\frac{1}{3\sigma^{4/3}}\Gamma(\frac{1}{3}) + \frac{1}{12}\sigma^{\frac{4}{3}}\Gamma(\frac{2}{3}) - \frac{1}{3^3\cdot 2^7}\sigma^{\frac{20}{3}}\Gamma(\frac{1}{3}) + ... \ ] \\ &= \frac{1}{6\sqrt{3}}[J_{\frac{1}{3}}(\frac{1}{12\sqrt{3}}\sigma^4) + J_{-\frac{1}{3}}(\frac{1}{12\sqrt{3}}\sigma^4)] \end{aligned} \tag{3.11}$$

This may also be written as an Airy function $Ai(z)$,

$$U(\sigma) = \frac{1}{3^{\frac{1}{3}}\sigma^{\frac{4}{3}}}Ai(-\frac{1}{4\cdot 3^{1/3}}\sigma^{\frac{8}{3}}) = \frac{1}{\sigma}\int_{-i\infty}^{i\infty}\frac{du}{2\pi i}e^{-\sigma u^3 - \frac{1}{4}\sigma^3 u} = \frac{1}{\sigma}\int_{-\infty}^{\infty}\frac{du}{2\pi}e^{i\sigma u^3 - \frac{i}{4}\sigma^3 u} \tag{3.12}$$

In deriving (3.11) we have made use of

$$\int_0^\infty \cos(t^3 - xt)dt = \frac{\pi}{3}\sqrt{\frac{x}{3}}[J_{1/3}(\frac{2x^{3/2}}{3^{3/2}}) + J_{-1/3}(\frac{2x^{3/2}}{3^{3/2}})] \tag{3.13}$$

The Airy function $Ai(z)$ was used for the case of two marked points for $p=3$ in [7].

We obtain thus the explicit expression for the intersection numbers

$$< \tau_{n,j} >_g = \frac{1}{(12)^g g!}\frac{\Gamma(\frac{g+1}{3})}{\Gamma(\frac{2-j}{3})} \tag{3.14}$$

with $n = (8g-5-j)/3$. This condition comes from general constraint for the intersection numbers of s-marked points of the moduli spaces of p-spin curves [28],

$$(p+1)(2g-2+s) = p\sum_{i=1}^s n_j + \sum_{i=1}^s j_i + s \tag{3.15}$$

The result of (3.14) agrees with (3.34) for $p=3$. We have

$$\begin{aligned} < \tau_{1,0} >_{g=1} &= \frac{1}{12}, \quad < \tau_{3,2} >_{g=2} = 0, \\ < \tau_{6,1} >_{g=3} &= \frac{1}{31104}, \quad < \tau_{9,0} >_{g=4} = \frac{1}{746496}, \ ..... \end{aligned} \tag{3.16}$$

(3) p=4

$$\begin{aligned} U(\sigma) &= \frac{1}{\sigma}e^{-\frac{1}{80}\sigma^5}\oint\frac{du}{2\pi i}e^{-\sigma u^4 - \frac{1}{2}\sigma^3 u^2} \\ &= \frac{1}{4\sigma^{5/4}}e^{-\frac{1}{80}\sigma^5}\oint\frac{dx}{2\pi i}x^{-\frac{3}{4}}e^{-x-\frac{1}{2}\sigma^{5/2}x^{1/2}} \\ &= \frac{\sin\frac{\pi}{2}}{4\pi\sigma^{5/2}}e^{-\frac{1}{80}\sigma^5}\int_0^\infty dx x^{-3/4}e^{-x+\frac{1}{2}\sigma^{5/2}x^{1/2}} \end{aligned} \tag{3.17}$$

where the contour integral reduces to tow integrals above and below the cut, as for the $p = 3$ case. We thereby obtain for $p = 4$,

$$
\begin{aligned}
U(\sigma) &= \frac{1}{4\pi}\sigma^{-5/4}e^{-\frac{1}{80}\sigma^5}\sum_{n=0}^{\infty}\frac{1}{n!}(\frac{\sigma^{5/2}}{2})^n\Gamma(\frac{n}{2}+\frac{1}{4}) \\
&= \frac{1}{4\pi}\sigma^{-5/4}e^{-\frac{1}{80}\sigma^5}[\Gamma(\frac{1}{4})+\frac{1}{2}\sigma^{5/2}\Gamma(\frac{3}{4})+\frac{1}{32}\sigma^5\Gamma(\frac{1}{4})+\cdots] \quad (3.18)
\end{aligned}
$$

This one-point function is the generating function

$$
U(\sigma) = \frac{1}{\pi}\sum_{n,j} < \tau_{n,j} >_g \sigma^{n+\frac{1+j}{4}} 4^{g-1}\Gamma(1-\frac{1+j}{4}) \quad (3.19)
$$

with $n = \frac{1}{4}(10g - 6 - j)$. Therefore the intersection numbers for $p = 4$ are

$$
\begin{aligned}
&< \tau_{1,0} >_{g=1} = \frac{1}{8}, \quad < \tau_{3,2} >_{g=2} = \frac{3}{2560}, \quad < \tau_{6,0} >_{g=3} = \frac{3}{20480}, \\
&< \tau_{8,2} >_{g=4} = \frac{77}{39321600}, \quad < \tau_{11,0} >_{g=5} = \frac{19}{104857600}, \ldots \quad (3.20)
\end{aligned}
$$

The exponent of the integrand (3.3) may be expressed as the Chebishev function $T_4(t,x) = t^4 + 4xt^2 + 2x^2$ with $x = \frac{1}{8}\sigma^{5/2}$, and we obtain a closed formula from the result of Appendix IV,

$$
U(\sigma) = \frac{1}{2\sqrt{8}}e^{\frac{3}{160}\sigma^5}\frac{1}{2\sin(\frac{\pi}{4})}[I_{-\frac{1}{4}}(\frac{1}{32}\sigma^5) + I_{\frac{1}{4}}(\frac{1}{32}\sigma^5)] \quad (3.21)
$$

Expanding above expression of the modified Bessel function $I_\nu(z)$ for small σ, we have

$$
\begin{aligned}
U(\sigma) &= \frac{1}{8}\sum_{m,n=0}^{\infty}\frac{1}{m!n!\Gamma(n+\frac{3}{4})}(\frac{3}{160})^m(\frac{1}{64})^{2n-\frac{1}{4}}\sigma^{5m+10n-\frac{1}{4}} \\
&+ \frac{1}{8}\sum_{m,n=0}^{\infty}\frac{1}{m!n!\Gamma(n+\frac{5}{4})}(\frac{3}{160})^m(\frac{1}{64})^{2n+\frac{1}{4}}\sigma^{5m+10n+\frac{1}{4}} \\
&= \frac{1}{8\pi}\Gamma(\frac{3}{4})\sigma^{\frac{5}{4}} + \frac{3}{640\pi}\Gamma(\frac{1}{4})\sigma^{\frac{15}{4}} + \cdots \quad (3.22)
\end{aligned}
$$

with $n = \frac{1}{4}(10g - 6 - j)$. which agrees with the previous results [6] and also it agrees with the result of Liu and Xu derived by the recursion formula from Gelfand-Dikii equation [35]. We have obtained a closed analytic formula for the intersection numbers of p=4 spin curves of one marked point in (3.21) for arbitrary genus g by expressing it as a Bessel function.

(4) $p = 5$

We have ,

$$\begin{aligned} U(\sigma) &= \frac{1}{\sigma}\oint \frac{du}{2\pi i} e^{-\frac{1}{6}[(u+\frac{\sigma}{2})^6-(u-\frac{\sigma}{2})^6]} \\ &= \frac{1}{\sigma}\oint \frac{du}{2\pi i} e^{-\sigma u^5-\frac{5}{6}\sigma^3 u^3-\frac{1}{16}\sigma^5 u} \\ &= \frac{1}{5\sigma^{6/5}}\oint \frac{dx}{2\pi i} x^{-4/5} e^{-x-\frac{5}{6}\sigma^{12/5}x^{3/5}-\frac{1}{16}\sigma^{24/5}x^{1/5}} \end{aligned} \quad (3.23)$$

By taking paths around a cut, similar to $p=3,4$ cases, we have

$$\begin{aligned} U(\sigma) &= \frac{1}{5i\sigma^{6/5}} e^{-\frac{8\pi i}{5}} \int_0^\infty \frac{dx}{2\pi} x^{-4/5} e^{-x\frac{5}{6}\sigma^{12/5}e^{6\pi i/5}x^{3/5}-\frac{1}{16}\sigma^{24/5}e^{2\pi i/5}x^{1/5}} \\ &- \frac{1}{5i\sigma^{6/5}} e^{\frac{8\pi i}{5}} \int_0^\infty \frac{dx}{2\pi} x^{-4/5} e^{-x\frac{5}{6}\sigma^{12/5}e^{-6\pi i/5}x^{3/5}-\frac{1}{16}\sigma^{24/5}e^{-2\pi i/5}x^{1/5}} \\ &= \frac{\sin\frac{2\pi}{5}}{5\pi}\Gamma(\frac{1}{5})\sigma^{-6/5} - \frac{\sin\frac{2\pi}{5}}{6\pi}\Gamma(\frac{4}{5})\sigma^{6/5} - \frac{11\sin(\frac{\pi}{5})}{720\pi}\Gamma(\frac{2}{5})\sigma^{18/5} \\ &+ \frac{\sin\frac{\pi}{5}}{\pi}\frac{341}{207360}\Gamma(\frac{3}{5})\sigma^{\frac{42}{5}} + \cdots \end{aligned} \quad (3.24)$$

We obtain

$$\begin{aligned} &< \tau_{1,0} >_{g=1} = \frac{1}{6}, \ < \tau_{3,2} >_{g=2} = \frac{11}{3600} \quad < \tau_{5,4} >_{g=3} = 0, \\ &< \tau_{8,1} >_{g=4} = \frac{341}{25920000}, \quad < \tau_{10,3} >_{g=5} = \frac{161}{777600000}, ... \end{aligned} \quad (3.25)$$

which agrees with [6] and [32].

We use $u = \sinh\theta$, and note that $T_5(iu) = i\cosh 5\theta$,

$$U(\sigma) = \sqrt{\frac{2}{3}}\int_0^\infty d\theta \cosh\theta \exp[-2x^{\frac{5}{2}}\cosh 5\theta + \frac{11\sqrt{2}}{16}\sigma^6 \sinh\theta] \quad (3.26)$$

with $x = \frac{1}{3}(\frac{1}{32})^{\frac{1}{5}}\sigma^{\frac{12}{5}}$. By the change of $\theta \to \frac{1}{5}\theta$, we have

$$U(\sigma) = \frac{1}{5}\sqrt{\frac{2}{3}}\int_0^\infty d\theta e^{-2x^{\frac{5}{2}}\cosh\theta} \sum_{n=0}^\infty \frac{1}{n!}(\frac{11\sqrt{2}}{16}\sigma^6 \sinh\frac{\theta}{5})^n \cosh\frac{\theta}{5} \quad (3.27)$$

This integral is evaluated by the formula,

$$\int_0^\infty d\theta e^{-z\cosh\theta-\nu\theta} = \frac{1}{\sin\nu\pi}\int_0^\pi d\theta e^{z\cos\theta}\cos\nu\theta - \frac{\pi}{\sin\nu\pi}I_\nu(z) \quad (3.28)$$

where $I_\nu(z)$ is modified Bessel function. The genus one ($g=1$) term of this series becomes

$$U(\sigma) \sim \frac{1}{5}\sqrt{\frac{2}{3}}K_{\frac{1}{5}}(\frac{1}{2\sqrt{2}\cdot 3^{\frac{5}{2}}}\sigma^6) \sim \frac{1}{6}\sigma^{\frac{6}{5}}\Gamma(1-\frac{1}{5}) + \cdots \quad (3.29)$$

which gives $\frac{1}{6}$ for the intersection numbers of the moduli space of $p=5$ spin curves.

We obtain from the equation of (3.27), the intersection numbers $<\tau_{n,j}>_g$, with condition $6(2g-1)=5n+j+1$, for $p=5$,

(5) general p

$$\begin{aligned} U(\sigma) &= \frac{1}{\sigma}\oint\frac{du}{2\pi i}e^{-\sigma u^p}\\ \times & \exp[-\frac{p(p-1)}{3!4}\sigma^3 u^{p-2}-\frac{p(p-1)(p-2)(p-3)}{5!4^2}\sigma^5 u^{p-4}-\cdots] \end{aligned} \tag{3.30}$$

By choosing a integral path around a cut,

$$\begin{aligned} U(\sigma) &= \mathrm{Re}\{\frac{e^{\frac{2\pi i}{p}}}{p\sigma^{1+\frac{1}{p}}\pi}\int_0^\infty dx x^{\frac{1}{p}-1}e^{-x}\exp[-\frac{p(p-1)}{3!4}\sigma^{2+\frac{2}{p}}e^{2\pi i(1-\frac{2}{p})}x^{1-\frac{2}{p}}\\ &- \frac{p(p-1)(p-2)(p-3)}{5!4^2}\sigma^{4+\frac{4}{p}}e^{2\pi i(1-\frac{4}{p})}x^{1-\frac{4}{p}}-\cdots]\} \end{aligned} \tag{3.31}$$

We have

$$\begin{aligned} U(\sigma) &= \frac{1}{\pi p\sigma^{1+\frac{1}{p}}}(\sin\frac{2\pi}{p})\Gamma(\frac{1}{p})+\frac{p-1}{24\pi}\sigma^{1+\frac{1}{p}}(\sin\frac{2\pi}{p})\Gamma(1-\frac{1}{p})\\ &- \frac{(p-1)(p-3)(2p+1)}{2760\pi}\sigma^{3+\frac{3}{p}}(\sin\frac{6\pi}{p})\Gamma(1-\frac{3}{p})\\ &- \frac{(p-1)(p-5)(1+2p)(8p^2-13p-13)}{7!4^3 3^2\pi}\sigma^{5+\frac{5}{p}}(\sin\frac{10\pi}{p})\Gamma(1-\frac{5}{p})\\ &+ \frac{(p-1)(p-7)(1+2p)(72p^4-298p^3-17p^2+562p+281)}{9!4^4 15}\sigma^{7+\frac{7}{p}}\\ &\times (\sin\frac{14\pi}{p})\Gamma(1-\frac{7}{p})+\cdots \end{aligned} \tag{3.32}$$

The intersection numbers of p spin curves are obtained with the condition $(p+1)(2g-1)=pn+j+1$.

$$U(\sigma)=\sum_g <\tau_{n,j}>_g p^{g-1}\sigma^{n+\frac{1+j}{p}}\Gamma(1-\frac{1+j}{p})\sin\frac{m}{p} \tag{3.33}$$

with $m=2\pi+4\pi(g-1)$.

$$\begin{aligned} &<\tau_{1,0}>_{g=1}=\frac{p-1}{24},\\ &<\tau_{n,j}>_{g=2}=\frac{(p-1)(p-3)(1+2p)}{p5!4^2 3}\frac{\Gamma(1-\frac{3}{p})}{\Gamma(1-\frac{1+j}{p})}, \end{aligned}$$

$$
\begin{aligned}
< \tau_{n,j} >_{g=3} &= \frac{(p-5)(p-1)(1+2p)(8p^2-13p-13)}{p^2 7! 4^3 3^2} \frac{\Gamma(1-\frac{5}{p})}{\Gamma(1-\frac{1+j}{p})} \\
< \tau_{n,j} >_{g=4} &= \frac{(p-1)(p-7)(1+2p)(72p^4-298p^3-17p^2+562p+281)}{p^3 9! 4^4 15} \\
&\times \frac{\Gamma(1-\frac{7}{p})}{\Gamma(1-\frac{1+j}{p})}
\end{aligned} \tag{3.34}
$$

This result is same as [6], where the integral is restricted to a path from 0 to ∞ without $\sin \frac{m}{p}$ factor in $U(\sigma)$.

(6) $p = -1$

This expression for arbitrary p in (3.34) allows the analytic continuation to the negative values of p. In the case $p = -1$, it correspond to Euler characteristics $\chi(M_{g,1}) = \zeta(1-2g)$ [6]. For $p = -1$, the power of $\sigma^{1+\frac{1}{p}}$ becomes zero, and σ dependence disappears. Therefore, we need the introduction of c, which is taken as N to specify the genus g.

$$
\begin{aligned}
U(\sigma) &= \frac{1}{N\sigma} \int \frac{du}{2\pi i} e^{-N \log \frac{u+\frac{1}{2}\sigma}{u-\frac{1}{2}\sigma}} \\
&= \frac{1}{N} \int \frac{du}{2\pi i} (\frac{u-\frac{1}{2}}{u+\frac{1}{2}})^N \\
&= \int_0^\infty dt \frac{1}{1-e^{-t}} e^{-Nt} = \sum_{n=1}^\infty (-1)^{n-1} \frac{B_n}{2nN^{2n}}
\end{aligned} \tag{3.35}
$$

We have used $\frac{u-1}{u+1} = e^{-y}$, and a following expansion,

$$
\frac{1}{1-e^{-t}} = \frac{1}{t} + \frac{1}{2} + \sum_{n=1}^\infty (-1)^{n-1} B_n \frac{t^{2n-1}}{(2n)!} \tag{3.36}
$$

This gives Euler characteristics (intersection number $< \tau >_g$ for $p = -1$),

$$
\chi(M_{g,1}) = < \tau >_g = -\frac{1}{2g} B_g = \zeta(1-2g) \tag{3.37}
$$

where ζ is the Riemann zeta function and B_n is a Bernoulli number ($B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $B_3 = \frac{1}{42}$, $B_4 = \frac{1}{30}$). When p is negative, we have to specify the meaning of spin j. This index p is related to the level k of the Lie group $su(2)_k/u(1)$. This was studied by Witten [36] as a chiral ring (Landau-Ginzburg theory) of primary fields and their gravitational descendants with

$$
p = k + 2 \tag{3.38}
$$

The present case corresponds to the singularity theory of A_{p-1}. When p is negative, we have a non-compact Lorenzian group $sl(2,R)_k/u(1)$, whose

discrete spectrum is known to correspond to two series $D^+_{\hat{k}}$ and $D^-_{\hat{k}}$ [37], The analytic continuation of $p \to -p$ corresponds to $D^-_{\hat{k}}$, and the spin j takes negative value. For instance, for $p = -1$, the Euler characteristics $\chi(M_{g,1})$ is defined by the top Chern class only and n , which is the power of the first Chern class, should be zero. Then we have only $< \tau_{0,-1} >_g$ in which $j = -1$.

$$< \tau_{n,j} >_g = \frac{1}{p^g} \int_{M_{g,1}} C_T(\nu)[c_1(\mathcal{L})]^n \tag{3.39}$$

Thus $< \tau_{n,-1} >_g$ is not surprising since the discrete spectrum with negative spin exists for $SL(2,R)$.

(7) $p = -2$

As noticed in [38], we have two expansions, weak coupling and strong coupling for $p = -2$, which correspond to the Gross-Witten model for the unitary group [27, 39]. There is a phase transition between these two phases. The weak coupling corresponds to small σ and strong coupling corresponds to large σ. Therefore, the spin values j takes negative values of $D^-_{\hat{k}}$ for weak coupling, and positive values for the strong coupling phase. The expansion of $u(\sigma)$ is expressed by putting $n = 0$ as

$$u(\sigma) = \sum a_j \sigma^{\frac{1+j}{p}} \tag{3.40}$$

More details of the discrete spectrum of $SL(2,R)_k$ are presented in appendix. Before closing this section on GUE, we write the one point function $U(\sigma)$ as an angular integral, which is useful for the strong coupling expansion.

In the expression of $U(\sigma)$ one puts $\sin\theta = 1/\sqrt{1+u^2}$ and $\cos\theta = u/\sqrt{1+u^2}$ in (3.3). Then with $\sigma = it$, and $u = \frac{t}{2}v$,

$$U(\sigma) = \frac{1}{2}\int \frac{dv}{2\pi} \exp[-\frac{c}{p+1}(\frac{t}{2})^{p+1}\{(v+i)^{p+1} - (v-i)^{p+1}\}] \tag{3.41}$$

With $v = \frac{\cos\theta}{\sin\theta}$, it becomes

$$U(\sigma) = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{d\theta}{2\pi} \frac{1}{(\sin\theta)^2} \exp[-\frac{2ic}{p+1}(\frac{t}{2})^{p+1}\frac{\sin(p+1)\theta}{(\sin\theta)^{p+1}}] \tag{3.42}$$

Note that the denominator of the exponent $(\sin\theta)^{p+1}$ becomes a numerator when $p+1$ is negative, and it provides a large σ expansion (strong coupling expansion for large t) corresponding to a discrete spectrum of $SL(2,R)_k$. This large $t = -i\sigma$ expansion becomes, for instance for $p = -2$,

$$\begin{aligned} U(\sigma) &= -\frac{1}{2}[D - (\frac{c}{t}) + (\frac{c}{t})^2 - (\frac{c}{t})^3 + \frac{5}{6}(\frac{c}{t})^4 - \frac{7}{12}(\frac{c}{t})^5 + \cdots] \\ &= -\frac{1}{2}\sum_m C_m(\frac{c}{t})^m \end{aligned} \tag{3.43}$$

with

$$C_m = \frac{(2m-1)!}{m!}\frac{1}{\prod_{l=1}^{m-1}(-l^2)} \tag{3.44}$$

D is a divergent term, which should be regularized. The above expression matches exactly a strong coupling expansion for the unitary (gauge) group, for a single trace result with $N = 0$ [38]. The unitary matrix model is

$$Z = \int dU e^{\mathrm{tr(UC^\dagger + U^\dagger C)}} \tag{3.45}$$

where U is a $N \times N$ unitary matrix, $UU^\dagger = 1$. C is an external complex matrix. The strong expansion is an expansion in powers of $\mathrm{tr(C^\dagger C)^m}$. The coefficient of $\mathrm{tr(C^\dagger C)^m}$, C_m is equal to

$$\begin{aligned} C_1 &= 1,\ C_2 = -\frac{1}{N^2-1},\ C_3 = \frac{4}{(N^2-1)(N^2-4)}, \\ C_4 &= -\frac{30}{(N^2-1)(N^2-4)(N^2-9)}, \cdots \end{aligned} \tag{3.46}$$

For obtaining the N dependence, we need the insertion of a logarithmic term in (3.3) as [38],

$$U(\sigma) = \frac{1}{2}\int \frac{du}{2i\pi} e^{\frac{4}{\sigma(u^2-1)}} (\frac{u-1}{u+1})^N \tag{3.47}$$

4 Classical Lie algebras

$\underline{X \in O(2N)}$

When the random matrix X varies over a classical Lie algebra, with Gaussian distributetion, the n-point correlation function in an external source is obtained again exactly , after use of the Harish Chandra formula[43]. We have discussed in earlier work such models with external source[12, 44].

Consider the Lie algebra of $O(2N)$, namely real antisymmetric matrices. Since, the Harish Chandra formula holds for this Lie algebra, we can obtain explicit expressions for the n-point correlation functions. Again one can derive a duality identity. In the present case, instead of the duality formula involving a supermatrix Q, it is convenient to use

$$< \prod_{\alpha=1}^{k} \det(\lambda_\alpha \cdot \mathrm{I} - X) >_A = < \prod_{n=1}^{N} \det(a_n \cdot \mathrm{I} - Y) >_\Lambda \tag{4.1}$$

where X is a $2N \times 2N$ real antisymmetric matrix ($X^t = -X$) and Y is $2k \times 2k$ real antisymmetric matrix ; the eigenvalues of X and Y are thus

pure imaginary. The matrix source A is also a $2N \times 2N$ antisymmetric matrix. The matrix Λ is q $2k \times 2k$ antisymmetric matrix, coupled to Y. We assume, without loss of generality, that A and Λ take the canonical form :

$$A = a_1 v \oplus \cdots \oplus a_N v, \quad v = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.2}$$

Λ is expressed also as

$$\Lambda = \lambda_1 v \oplus \cdots \oplus \lambda_k v. \tag{4.3}$$

The definition of the averages are

$$\langle \mathcal{O}(X) \rangle_A = \frac{1}{Z_A} \int dX \ \mathcal{O}(X) \ \exp(\frac{1}{2}\mathrm{trX}^2 + \mathrm{trAX}) \tag{4.4}$$

$$\langle \mathcal{O}(Y) \rangle_\Lambda = \frac{1}{Z_\Lambda} \int dY \ \mathcal{O}(Y) \ \exp(\frac{1}{2}\mathrm{trY}^2 + \mathrm{tr\Lambda Y}) \tag{4.5}$$

By an appropriate tuning of the a_n's, and a corresponding rescaling of Y and Λ, one may generate similarly higher models of type p with the conditions (2.11),

$$Z = \int dY e^{-\frac{1}{p+1}\mathrm{tr}Y^{p+1} + \mathrm{tr}Y\Lambda} \tag{4.6}$$

where p is an odd integer.

The HarishChandra integral for the integral over $g \in SO(2N)$ group, and given real antisymmetric matrices Y and Λ, reads

$$\int_{SO(2N)} dg e^{\mathrm{tr}(gYg^{-1}\Lambda)} = C \frac{\sum_{w\in W} (\det w) \exp[2 \sum_{j=1}^{N} w(y_j)\lambda_j]}{\prod_{1\le j<k\le N} (y_j^2 - y_k^2)(\lambda_j^2 - \lambda_k^2)} \tag{4.7}$$

where $C = (2N-1)! \prod_{j=1}^{2N-1} (2j-1)!$, and w are elements of the Weyl group, which consists here of permutations followed by reflections ($y_i \to \pm y_i$; $i = 1, \cdots, N$) with an even number of sign changes.

For the one point function, we obtain when X is a $2N \times 2N$ real antisymmetric random matrix, from the above formula,

$$\begin{aligned} U(\sigma) &= \frac{1}{2N} < \mathrm{tr} e^{\sigma X} >_A \\ &= \frac{1}{2N} \sum_{\alpha=1}^{N} \prod_{\gamma\neq\alpha}^{N} (\frac{(a_\alpha + \frac{\sigma}{2})^2 - a_\gamma^2}{a_\alpha^2 - a_\gamma^2}) e^{\sigma a_\alpha + \frac{\sigma^2}{4}} + (\sigma \to -\sigma) \\ &= \frac{1}{N\sigma} \oint \frac{du}{2\pi i} (\frac{(u + \frac{\sigma}{2})^2 - a_\gamma^2}{u^2 - a_\gamma^2}) \frac{u}{u + \frac{\sigma}{4}} e^{\sigma u + \frac{\sigma^2}{4}} \end{aligned} \tag{4.8}$$

where the contour encircles the poles $u = a_\gamma$. Or, shifting $u \to u - \frac{\sigma}{4}$,

$$U(\sigma) = \frac{1}{N\sigma} \oint \frac{dv}{2\pi i} \prod_{i=1}^{N} \frac{(u - \frac{\sigma}{4})^2 - a_i^2}{(u + \frac{\sigma}{4})^2 - a_i^2} \left(\frac{u - \frac{\sigma}{4}}{u} \right) e^{\sigma u} \tag{4.9}$$

Tuning the external source to obtain the p-th degeneracy, one finds

$$U(\sigma) = \frac{1}{N\sigma} \oint \frac{du}{2i\pi} e^{-\frac{c}{p+1}[(u+\frac{\sigma}{4})^{p+1} - (u-\frac{\sigma}{4})^{p+1}]} (1 - \frac{\sigma}{4u}) \tag{4.10}$$

(1) $p = 3$

There are two terms in (4.10) ; the first term $U(\sigma)^{OR}$ is exactly one-half of $U(\frac{\sigma}{2})$ for the GUE (orientable Riemann surfaces). The second term is a new term, and we denote it as the non-orientable part $\mathrm{U}(\sigma)^{NO}$, since it is related to non-orientable surfaces with half-integer genus :

$$\begin{aligned} U(\sigma)^{OR} &= \frac{1}{2} U(\frac{\sigma}{2}) \\ &= \frac{1}{12\sqrt{3}} [J_{\frac{1}{3}}(\frac{1}{12\sqrt{3}}(\frac{\sigma}{2})^4) + J_{-\frac{1}{3}}(\frac{1}{12\sqrt{3}}(\frac{\sigma}{2})^4)] \\ &= \frac{1}{2 \cdot 3^{\frac{1}{3}} (\frac{\sigma}{2})^{\frac{4}{3}}} Ai(-\frac{1}{4 \cdot 3^{1/3}} (\frac{\sigma}{2})^{\frac{8}{3}}) \end{aligned} \tag{4.11}$$

For the non-orientable surfaces, from the condition,

$$(p+1)(2g-1) = pn + j + 1 \tag{4.12}$$

we find that the genus g is always a half-integer ($g = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, ...$), and $\mathrm{U}(\sigma)^{NO}$ has a series expansion in powers of $\sigma^{n+\frac{1+j}{p}}$. For $p = 3$, we have

$$\begin{aligned} U(\sigma)^{NO} &= \frac{1}{4} \oint \frac{du}{2\pi i} \frac{1}{u} e^{-\frac{\sigma}{2}u^3 - \frac{\sigma^3}{32}u} \\ &= \frac{1}{12} \oint \frac{dx}{2\pi i} \frac{1}{x} e^{-x - \frac{2^{1/3}}{32}\sigma^{8/3}x^{1/3}} \\ &= \mathrm{Re}\{\frac{1}{12 i\pi} \int_0^\infty \frac{1}{x} e^{-x - \frac{1}{4} e^{\frac{2\pi i}{3}} (\frac{\sigma}{2})^{8/3} x^{1/3}} \} \end{aligned} \tag{4.13}$$

This function may be expanded as

$$\begin{aligned} U(\sigma)^{NO} &= \mathrm{Re}\{\frac{1}{12 i\pi} \int_0^\infty dx \frac{1}{x} e^{-x} \sum_{n=0}^{\infty} \frac{1}{n!} (-\frac{1}{4} e^{\frac{2\pi i}{3}} (\frac{\sigma}{2})^{8/3} x^{1/3})^n \} \\ &= -\frac{1}{\pi} \frac{1}{48} (\frac{\sigma}{2})^{\frac{8}{3}} (\sin \frac{2\pi}{3}) \Gamma(\frac{1}{3}) \\ &+ \frac{1}{\pi} \frac{1}{384} (\frac{\sigma}{2})^{\frac{16}{3}} (\sin \frac{4\pi}{3}) \Gamma(\frac{2}{3}) - \frac{1}{\pi} \frac{1}{3! \cdot 12 \cdot 4^3} (\frac{\sigma}{2})^8 (\sin 2\pi) \\ &+ \frac{1}{\pi} \frac{1}{4! \cdot 12 \cdot 4^4} (\frac{\sigma}{2})^{\frac{32}{3}} (\sin \frac{8\pi}{3}) \Gamma(\frac{4}{3}) - \cdots \end{aligned} \tag{4.14}$$

Using Airy functions, the $p = 3$ case is expressed as

$$U(\sigma) = \frac{1}{2 \cdot 3^{1/3}(\frac{\sigma}{2})^{4/3}} Ai(x) - \frac{1}{4}\int_0^x dx' Ai(x') \tag{4.15}$$

with $x = -\frac{1}{4 \cdot 3^{1/3}}(\frac{\sigma}{2})^{8/3}$.

The Airy function $Ai(z)$ and the integral of Airy function may be expanded as

$$Ai(z) = \frac{\pi}{3^{2/3}} \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+\frac{2}{3})} (\frac{1}{3})^{2n} z^{3n} - \frac{\pi}{3^{4/3}} \sum_{0}^{\infty} \frac{1}{n!\Gamma(n+\frac{4}{3})} (\frac{1}{3})^{2n} z^{3n+1} \tag{4.16}$$

$$\int_0^z Ai(t)dt = \frac{\pi}{3^{2/3}\Gamma(\frac{2}{3})} z - \frac{\pi}{3^{4/3} \cdot 2\Gamma(\frac{4}{3})} z^2 + \frac{\pi}{36 \cdot 3^{2/3}\Gamma(\frac{5}{3})} z^4 + \cdots \tag{4.17}$$

Inserting these expansions, we have for $p = 3$, $\frac{\pi}{\sin(\frac{\pi}{3})} = \Gamma(\frac{1}{3})\Gamma(\frac{2}{3})$.

$$\begin{aligned} U(\sigma) &= \frac{\pi}{24\Gamma(\frac{1}{3})}(\frac{\sigma}{2})^{4/3} - \frac{\pi}{108 \cdot 64\Gamma(\frac{2}{3})}(\frac{\sigma}{2})^{20/3} + \cdots \\ &+ [\frac{\pi}{48\Gamma(\frac{2}{3})}(\frac{\sigma}{2})^{8/3} + \frac{\pi}{384\Gamma(\frac{1}{3})}(\frac{\sigma}{2})^{16/3} - \frac{\pi}{864 \cdot 4^4\Gamma(\frac{2}{3})}(\frac{\sigma}{2})^{32/3} + \cdots] \end{aligned} \tag{4.18}$$

$$\begin{aligned} U(\sigma) &= <\tau_{1,0}>_{g=1} \Gamma(1-\frac{1}{3})(\frac{\sigma}{2})^{1+\frac{1}{3}} + <\tau_{6,1}>_{g=3} \Gamma(1-\frac{2}{3})3^2(\frac{\sigma}{2})^{6+\frac{2}{3}} + \cdots \\ &+ [<\tau_{2,1}>_{g=3/2} \Gamma(1-\frac{2}{3})3^2(\frac{\sigma}{2})^{2+\frac{2}{3}} + <\tau_{5,0}>_{g=5/2} \Gamma(1-\frac{1}{3})3^4(\frac{\sigma}{2})^{16/3} \\ &+ <\tau_{10,1}>_{g=9/2} \Gamma(1-\frac{2}{3})3^8(\frac{\sigma}{2})^{32/3} + \cdots] \end{aligned} \tag{4.19}$$

We have for $p = 3$,

$$\begin{aligned} U(\sigma)^{NO} &= \frac{1}{12}y^2\Gamma(1-\frac{2}{3}) + \frac{1}{24}y^4\Gamma(1-\frac{1}{3}) \\ &+ \frac{1}{864}y^8\Gamma(1-\frac{2}{3}) + ... \end{aligned} \tag{4.20}$$

We have obtained for $p = 3$ the explicit intersection numbers for non-orientable surfaces with one marked point. The intersection number $<\tau_{2,1}>_{g=3/2}$ corresponds to a cross-capped torus. For $g = 1/2$ wa are dealing with the topology of the projective plane but for this case, the intersection numbers $<\tau_{0,1}^2>_{g=1/2}$ are present only beyond the two marked points level [12]. We have

$$<\tau_{1,0}>_{g=1} = \frac{1}{24}, \ <\tau_{2,1}>_{g=\frac{3}{2}} = \frac{1}{864}, \ ... \tag{4.21}$$

(2) general p

Using the binomial expansion, one finds ($y = 2^{\frac{1}{p}}(\frac{\sigma}{4})^{1+\frac{1}{p}} = \frac{1}{2}(\frac{\sigma}{2})^{1+\frac{1}{p}}$)

$$\begin{aligned} U(\sigma) = \quad & -\frac{1}{4ypN}\int dt t^{\frac{1}{p}-1}e^{-t}[1-\frac{p(p-1)}{6}y^2t^{1-\frac{2}{p}}+\cdots] \\ & \times[1+yt^{-\frac{1}{p}}] \end{aligned} \tag{4.22}$$

This is again the sum of two contributions, orientable (OR) and non-orientable (NO). The odd powers in y correspond to the orientable contribution, which is the same as for the unitary case ; the even powers in y correspond to the non-orientable case :

$$U(\sigma) = U(\sigma)^{OR} + U(\sigma)^{NO} \tag{4.23}$$

$U(\sigma)^{OR}$ is same as GUE but the normalization of σ is replaced by $\sigma/2$.

The first term in the above series expansion is divergent, and it should be regularized. Except for this divergent term, we give the series expansion up to order y^8 (we have neglected the phase factor $\sin(\frac{2\pi m}{p})$,

$$\begin{aligned} U(\sigma)^{NO} &= \frac{y^2}{24}(p-1)\Gamma(1-\frac{2}{p}) \\ &+ \frac{y^4}{6!}(p-1)(p^2-5p+1)\Gamma(1-\frac{4}{p}) \\ &+ \frac{y^6}{7!\cdot 9}(p-1)(p-3)(4p^3-23p^2-2p-6)\Gamma(1-\frac{6}{p}) \\ &+ \frac{y^8}{7!3^3\cdot 10}(p-1)(9p^6-121p^5+435p^4-317p^3 \\ & -167p^2-471p-43)\Gamma(1-\frac{8}{p}) + O(y^{10}) \end{aligned} \tag{4.24}$$

From this genus expansion, one obtains the intersection numbers of p-spin curves for non-orientable surfaces.

(3) $p = -1$

We now perform the limit, $p \to -1$, which is related to the virtual Euler characteristics. When we put $p = -1$ in (4.24), the Γ function term becomes an integer for $p = -1$, and this agrees with the intersection number of $< \tau_{1,0} >_g$, which gives a factor $\Gamma(1-\frac{1}{p}) = \Gamma(2) = 1$ for the spin zero. We obtain

$$U(\sigma)^{NO} = -\frac{1}{24}(2y)^2 - \frac{7}{240}(2y)^4 - \frac{31}{504}(2y)^6 - \frac{127}{480}(2y)^8 + \cdots \tag{4.25}$$

This series agrees precisely with the series expansion

$$U(\sigma)^{NO} = -\sum_{\hat{g}=1}^{\infty} \frac{1}{2\hat{g}}(2^{2\hat{g}-2} - \frac{1}{2})B_{\hat{g}}(2y)^{2\hat{g}} \tag{4.26}$$

where $B_{\hat{g}}$ is a Bernoulli number, a positive rational number. $B_1 = \frac{1}{6}, B_2 = \frac{1}{30}, B_3 = \frac{1}{42}, B_4 = \frac{1}{30}$. The coefficient of $(2y)^{2\hat{g}}$ is the same as for the virtual Euler characteristics of the moduli space of real algebraic curves for genus g and one marked point, which was derived from the Penner model of the real symmetric matrix by Goulden et al. [42]. (We use for the half genuses in the list , $\frac{1}{2}$, 1, $\frac{3}{2}$, 2,... for a projective plane, Klein bottle, cross-capped torus, doubly cross-capped torus ,..., with the notation $\hat{g} = 1$, $\hat{g} = 2$, $\hat{g} = 3$,$\hat{g} = 4$,....,respectively [41], and this is a reason for the appearance of the $(2y)^{2\hat{g}}$ factor in (4.25)).

Since we derived this from the antisymmetric $O(2N)$ Lie algebra, the coincidence between $O(2N)$ lie algebra and GOE for the virtual Euler characteristics seems remarkable.

$$\chi^{NO}(\bar{M}_{g,1}) = \frac{1}{2g}(\frac{1}{2} - 2^{2g-2})B_g. \tag{4.27}$$

This result may be obtained analytically to all orders. We now derive this result from the integral form (4.10) replacing c by N. With $p = -1$, it becomes

$$U(\sigma) = -\frac{1}{4N\sigma}\int du(\frac{u-\sigma}{u+\sigma})^N(1+\frac{\sigma}{u}) \tag{4.28}$$

With the change of variable $u \to \sigma u$,

$$U(s) = -\frac{1}{4N}\int du(\frac{u-1}{u+1})^N(1+\frac{1}{u}) \tag{4.29}$$

We divide it into two parts, $U(\sigma)^{OR}$ and $U(\sigma)^{NO}$,

$$U(\sigma)^{OR} = -\frac{1}{4N}\int du(\frac{u-1}{u+1})^N \tag{4.30}$$

$$U(\sigma)^{NO} = -\frac{1}{4N}\int du(\frac{u-1}{u+1})^N\frac{1}{u} \tag{4.31}$$

We use the same change of variables as for the unitary case [4],

$$\frac{u-1}{u+1} = e^{-y}, \quad u = \frac{1+e^{-y}}{1-e^{-y}}, \quad du = -2\frac{e^{-y}}{(1-e^{-y})^2}dy \tag{4.32}$$

$$U(\sigma)^{OR} = \frac{1}{2N}\int dy e^{-Ny}\frac{e^{-y}}{(1-e^{-y})^2} \tag{4.33}$$

$$
\begin{aligned}
U(\sigma)^{NO} &= \frac{1}{2N}\int dye^{-Ny}\frac{e^{-y}}{(1-e^{-y})^2}(\frac{1-e^{-y}}{1+e^{-y}}) \\
&= \frac{1}{4N}\int dye^{-Ny}[\frac{1}{1-e^{-y}}-\frac{1}{1+e^{-y}}]
\end{aligned}
\tag{4.34}
$$

It is interesting to note that both Boson and Fermion distributions enter in the above integrand (4.34).

If we use the expansions,

$$
\begin{aligned}
\frac{1}{1-e^{-y}} &= \frac{1}{y}+\frac{1}{2}+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{B_n}{(2n)!}y^{2n-1} \\
\frac{1}{1+e^{-y}} &= \frac{1}{2}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2^{2n}-1)}{(2n)!}B_n y^{2n-1}
\end{aligned}
\tag{4.35}
$$

then they become

$$
\begin{aligned}
U(\sigma)^{OR} &= \frac{1}{2N}\int dy\frac{1}{y^2}e^{-Ny}-\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{B_n}{2n}\frac{1}{N^{2n}} \\
U(\sigma)^{NO} &= \frac{1}{4N}\int dye^{-Ny}+\frac{1}{4}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{B_n}{2n}\frac{1}{N^{2n+1}} \\
&- \frac{1}{4}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(2^{2n}-1)}{2n}B_n\frac{1}{N^{2n+1}} \\
&= \frac{1}{4N}\int dy\frac{e^{-Ny}}{y}+\frac{1}{4}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(2-2^{2n})B_n}{2n}\frac{1}{N^{2n+1}}
\end{aligned}
\tag{4.36}
$$

We now get from the above equation (replacing n by g),

$$
\begin{aligned}
\chi^{OR}(\bar{M}_{g,1}) &= -\frac{1}{2}\zeta(1-2g) = -\frac{1}{2}\frac{(-1)^g B_g}{2g}, \\
\chi^{NO}(\bar{M}_{g,1}) &= (-1)^{g-1}\frac{1}{2g}(2^{2g-2}-2^{-1})B_g
\end{aligned}
\tag{4.37}
$$

For s marked point, the result obtained from the real symmetric matrix Penner model [42] is

$$
\chi^{NO}(\bar{M}_{g,s}) = (-1)^s\frac{1}{2}\frac{(2g+s-2)!(2^{2g-1}-1)}{(2g)!s!}B_g \tag{4.38}
$$

This result can be obtained by applying equation (4.37) [7]. In this O(2N) model, we have the following condition, the same as for Riemann surfaces with spin j and s-marked points

$$
(p+1)(2g-2+s) = p\sum_{i=1}^{s}n_i+\sum_{i=1}^{s}j_i+s \tag{4.39}
$$

However, we have to assign the genus g also to half integers to represent non-orientable surfaces [12].

$\underline{X \in O(2N+1)}$

For $SO(2N+1)$ Lie algebra, the matrix X is

$$X = h_1 v \oplus h_2 v \oplus \cdots h_N v \oplus 0 \tag{4.40}$$

The measure is $V(H)^2$,

$$V(H) = \prod_{1 \le j \le N} (h_j^2 - h_k^2) \prod_{j=1}^{N} h_j \tag{4.41}$$

The Harish Chandra formula is

$$I = \int_{SO(2N+1)} e^{\mathrm{tr}(gag^{-1}b)} dg = C_{G(N)} \frac{\sum\limits_{w \in G(N)} (\det w) \exp(2 \sum\limits_{j=1}^{N} w(a_j) b_j)}{\prod\limits_{1 \le j \le k \le N} (a_j^2 - a_k^2)(b_j^2 - b_k^2) \prod\limits_{j=1}^{N} a_j b_j} \tag{4.42}$$

with $C_{G(n)} = \prod_{j=1}^{N} (2j-1)! \prod_{j=2N}^{4N-1} j!$. Comparing with the $O(2N)$ case, this formula differs from (4.7) by the presence of the term $\prod a_j b_j$ in the denominator. For the one point function, we have

$$U(\sigma) = \frac{1}{N} \sum_{\alpha=1}^{N} \int_{-\infty}^{\infty} \prod_{i=1}^{N} d\lambda_i \frac{\prod (\lambda_i^2 - \lambda_j^2) \prod \lambda_k}{\prod (a_i^2 - a_j^2) \prod a_k} e^{-\sum \lambda_i^2 + \sigma \lambda_\alpha + 2 \sum a_i \lambda_i} \tag{4.43}$$

This sum of integrals may be written as a contour integral, which collects poles at $u = a_i^2$,

$$\begin{aligned} U(\sigma) &= \oint_{\{u=a_i^2\}} \frac{du}{2\pi i} \prod_{j=1}^{N} \frac{(\sqrt{u}+\sigma)^2 - a_j^2}{u - a_j^2} \frac{1}{(\sqrt{u}+\sigma)^2 - u} (1 + \frac{\sigma}{\sqrt{u}}) e^{\sigma^2 + 2\sigma\sqrt{u}} \\ &= \frac{2}{\sigma} \oint \frac{dv}{2\pi i} \prod_{j=1}^{N} \frac{(v+\sigma)^2 - a_j^2}{v^2 - a_j^2} \frac{v+\sigma}{\sigma + 2v} e^{\sigma^2 + 2\sigma v} \\ &= \frac{1}{\sigma} \oint \frac{dv}{2\pi i} \prod_{j=1}^{N} \frac{(v + \frac{\sigma}{2})^2 - a_j^2}{(v - \frac{\sigma}{2})^2 - a_j^2} (1 + \frac{\sigma}{2v}) e^{\sigma v} \end{aligned} \tag{4.44}$$

By the tuning to the p-th degeneracy, we obtain

$$U(\sigma) = \frac{1}{\sigma} \oint \frac{du}{2\pi i} e^{-\frac{1}{p+1}((u+\frac{\sigma}{2})^{p+1} - (u - \frac{\sigma}{2})^{p+1})} (1 + \frac{\sigma}{2u}) \tag{4.45}$$

This takes the same form as for the $O(2N)$ case.

$\underline{X \in Sp(N)}$

The Haar measure of $Sp(N)$ is $\Delta(\lambda)^2$, with

$$\Delta(\lambda) = \prod_{i<j}(\lambda_i^2 - \lambda_j^2)\prod_k \lambda_k \tag{4.46}$$

The Harish Chandra formula for $Sp(N)$ reads [44]

$$\begin{aligned} I &= \int_G e^{<Ad(g)\cdot a|b>}dg = \frac{\sum_{w\in W}(\mathrm{det}w)e^{<w\cdot a|b>}}{\Delta(a)\Delta(b)} \\ &= C\frac{\det[2\sinh(2a_ib_j)]}{\prod(a_i^2-a_j^2)(b_i^2-b_j^2)\prod(a_kb_k)} \end{aligned} \tag{4.47}$$

For the one point function, we have

$$\begin{aligned} U(\sigma) &= \frac{1}{N}\sum_{\alpha=1}^N\int_{-\infty}^{\infty}\prod_{i=1}^N d\lambda_i \frac{\prod\limits_{1\le i<j\le N}(\lambda_i^2-\lambda_j^2)\prod\limits_{1\le k\le N}\lambda_k}{\prod\limits_{1\le i<j\le N}(a_i^2-a_j^2)\prod\limits_{1\le k\le N}a_k}e^{-\sum\lambda_i^2+\sigma\lambda_\alpha+2\sum a_i\lambda_i} \\ &= \oint\frac{du}{2\pi i}\prod_{j=1}^N\frac{(\sqrt{u}+\sigma)^2-a_j^2}{u-a_j^2}\frac{1}{(\sqrt{u}+\sigma)^2-u}(1+\frac{\sigma}{\sqrt{u}})e^{\sigma^2+2\sigma\sqrt{u}} \\ &= \frac{2}{\sigma}\oint\frac{dv}{2\pi i}\prod_{j=1}^N\frac{(v+\sigma)^2-a_j^2}{v^2-a_j^2}\frac{v+\sigma}{\sigma+2v}e^{\sigma^2+2\sigma v} \\ &= \frac{1}{\sigma}\oint\frac{dv}{2\pi i}\prod_{j=1}^N\frac{(v+\frac{\sigma}{2})^2-a_j^2}{(v-\frac{\sigma}{2})^2-a_j^2}(1+\frac{\sigma}{2v})e^{\sigma v} \end{aligned} \tag{4.48}$$

where we have shifted $v\to v-\frac{\sigma}{2}$ and $a_\gamma\to a_\gamma/2$. This expression becomes the same as for the $O(2N)$ case, when we put $v\to 2v$ up to a factor 2. Note that we do not need to consider the expansion $\frac{\sigma}{2}$ as in the $O(2N)$ case. The first term of the expression is same as for GUE. By the tuning a_γ to the p-th case, we have

$$U(\sigma) = \frac{1}{\sigma}\oint\frac{du}{2\pi i}e^{-\frac{1}{p+1}((u+\frac{\sigma}{2})^{p+1}-(u-\frac{\sigma}{2})^{p+1})}(1+\frac{\sigma}{2u}) \tag{4.49}$$

We write these two terms as $U(\sigma) = U(\sigma)^{OR}+U(\sigma)^{NO}$. It is then obvious that we obtain the same intersection numbers and virtual Euler characteristics as in the $O(2N)$ case.

5 Open intersection numbers

Kontsevich-Penner model

The Airy matrix model with an external source, the Kontsevich model for $p = 2$, gives the intersection numbers for closed Riemann surfaces, which satisfy a KdV hierarchy. These closed intersection numbers are obtained from (3.33) and (3.34) for one marked point. They are known for genus g and one marked point in a simple closed form,

$$< \tau_{3g-2,0} >_g = \frac{1}{(24)^g g!} \tag{5.1}$$

When the Riemann surface has boundaries, open intersection numbers appear, which differ from that of the Kontsevich model. We have studied earlier the effect of an additional logarithmic potential in the Kontsevich model, the so called Kontsevich-Penner model [19]. In our work this model came from a two matrix model, which originated itself from a time-dependent matrix model. The eigenvalues of the two matrices correspond for one to the edge of the distribution and for the other one to the bulk. Then we can use the duality identity for the two characteristic polynomials of the two matrices with external sources, and thereby recover the Kontsevich-Penner model. Therefore the presence in that model of the term i of $(\det M)^k = \exp[k\mathrm{trlogM}]$ coreesponds to the addition of a boundary (an open disc) in the random surfaces described by the Kontsevich model. Recently the open intersection numbers have been analyzed in [16, 17, 18]. The generating matrix model for those open intersection numbers are given by a Kontsevich-Penner model[21, 22]. This Kontsevich-Penner model has different Virasoro equations and different intersection numbers, which depend upon an additional parameter k which corresponds to the logarithmic term

$$Z = \int dM e^{\frac{1}{3}\mathrm{trM}^3 + \mathrm{trM}\Lambda + \mathrm{ktrlogM}} \tag{5.2}$$

For the open intersection numbers, considered by [16], k takes the value $k = 1$ [22]. The addition of the logarithmic potential yields new Virasoro equations and new intersection numbers which related to the boundary insertions. The intersection numbers for the model (5.2) have been computed in [19],

$$< \tau_1 >_{g=1} = \frac{1 + 12k^2}{24}, \quad < \tau_0 \tau_{\frac{1}{2}} >_{g=\frac{1}{2}} = k, \quad ... \tag{5.3}$$

The appearance of a half-integer index exhibits the non-orientable nature. The non-vanishing $< \tau_{n_1} \cdots \tau_{n_s} >_g$ are restricted by the condition

$$3(2g - 2 + s) = 2 \sum_{i=1}^{s} n_i + s \tag{5.4}$$

When the parameter k vanshes, the intersection numbers reduce to the usual Kontsevich result, which satisfies a KdV hierarchy. When the cubic Airy

matrix part is absent, and only the logarithmic potential is present (Penner model), as we have seen in the $p = -1$ case in section 3, the model gives the Euler characterstics [6]. When $k = 1$, it reduces to open intersection numbers. The meaning of the parameter k is found in the two matrix model [6, 19].

We now consider the k-dependence with one marked point. The one point intersection numbers of the Kontsevich-Penner model (5.2) are obtained from $U(\sigma)$ [19]

$$\begin{aligned} U(\sigma) &= \frac{1}{\sigma}\oint \frac{du}{2\pi i} e^{-\frac{c}{3}[(u+\frac{\sigma}{2})^3-(u-\frac{\sigma}{2})^3]+k\log(u+\frac{\sigma}{2})-k\log(u-\frac{\sigma}{2})} \\ &= \frac{1}{\sigma}e^{-\frac{c}{12}\sigma^3}\oint \frac{du}{2\pi i} e^{-c\sigma u^2+k\log(u+\frac{1}{2}\sigma)-k\log(u-\frac{1}{2}\sigma)} \end{aligned} \tag{5.5}$$

with

$$\sigma = \frac{1}{\lambda}, \quad t_n = \frac{1}{\lambda^{n+\frac{1}{2}}} \tag{5.6}$$

This $U(\sigma)$ correctly reduces to the intersection numbers of the Kontsevich model with one marked point when $k = 0$,

$$U(\sigma) = \frac{\sqrt{\pi}}{\sqrt{c}}\sum_{g=1}^{\infty}\frac{(-c)^g}{(12)^g g!}t_{3g-2} \tag{5.7}$$

Including the factor $\frac{1}{(-cp)^g}$, (p=2), the intersection number reduces to

$$< \tau_{3g-2} >_g = \frac{1}{(24)^g g!} \tag{5.8}$$

For dealing with higher k's, we expand (5.5), after rescaling of u,

$$\begin{aligned} U(\sigma) &= \frac{1}{2\sigma^{\frac{3}{2}}}e^{-\frac{c\sigma^3}{12}}\oint\frac{du}{2\pi i}e^{-\frac{c}{4}u^2}[1+k(\frac{2}{u}\sigma^{\frac{3}{2}}+\frac{2}{3u^3}\sigma^{\frac{9}{2}}+\frac{2}{5u^5}\sigma^{\frac{15}{2}}\cdots) \\ &+ k^2(\frac{2}{u^2}\sigma^3+\frac{4}{3u^4}\sigma^6+\cdots) \\ &+ k^3(\frac{8}{3!u^3}\sigma^{\frac{9}{2}}+\cdots)+k^4(\frac{16}{4!u^4}\sigma^6+\cdots)+O(k^5)] \end{aligned} \tag{5.9}$$

The coefficients of the successive orders in k tmay be computed from

$$\begin{aligned} & k\frac{e^{-\frac{c}{12}\sigma^3}}{\sigma^{\frac{3}{2}}}\oint\frac{du}{2\pi i}e^{-\frac{c}{4}u^2}\log\frac{u+\sigma^{3/2}}{u-\sigma^{3/2}} \\ &= k\frac{1}{\sigma^{\frac{3}{2}}}\sqrt{\frac{\pi}{c}}e^{-\frac{c}{12}\sigma^3}\mathrm{erf}(\frac{\sqrt{c}}{2}\sigma^{3/2}) \end{aligned} \tag{5.10}$$

with

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \tag{5.11}$$

where the integral is computed as the discontinuity across the cut between -1 to 1 in the u-plane. This integral becomes a contour integral around $u = 0$ by expanding the logarithm in powers of $\frac{1}{u}$ as

$$\begin{aligned} &\frac{k}{\sigma^{3/2}} e^{-\frac{c\sigma^3}{12}} \oint \frac{du}{2\pi i} e^{-\frac{c}{4}u^2} \sum_{n=0}^{\infty} \frac{\sigma^{\frac{3}{2}(2n+1)}}{(2n+1)u^{2n+1}} \\ &= k e^{-\frac{c}{12}\sigma^3} \sum_{n=0}^{\infty} \frac{1}{n!(2n+1)} (-\frac{c\sigma^3}{4})^n \end{aligned} \tag{5.12}$$

For the odd powers of k, the integration over u is the same contour integral around $u = 0$ [19].

For the even powers of k, we use the following integrals,

$$\int_{-\infty}^{\infty} du e^{-au^2} \frac{1}{u^{2n}} = (-1)^n \frac{2^n \sqrt{\pi}}{(2n-1)!!} a^{\frac{2n-1}{2}} \tag{5.13}$$

which may be obtained by integration over a. Putting $a = \frac{c}{4}$ we obtain

$$\int e^{-\frac{c}{4}u^2} \frac{1}{u^2} du = -\sqrt{c\pi} \tag{5.14}$$

Thus we obtain, up to terms of order $k^2\sigma^3$,

$$\begin{aligned} U(\sigma) &= e^{-\frac{c\sigma^3}{12}} \frac{1}{2\sigma^{\frac{3}{2}}} \int \frac{du}{2\pi} e^{-\frac{c}{4}u^2} [1 + 2k^2\sigma^3 \frac{1}{u^2}] \\ &= \frac{1}{2\pi} \sqrt{\frac{\pi}{c}} e^{-\frac{c}{12}\sigma^3} \frac{1}{\sigma^{\frac{3}{2}}} (1 - ck^2\sigma^3) \end{aligned} \tag{5.15}$$

Expanding the factor $e^{-\frac{c}{12}\sigma^3}$, we obtain the intersection number $< \tau_1 >$ as

$$< \tau_1 >_{g=1} = \frac{1}{24}(1 + 12k^2) \tag{5.16}$$

For $< \tau_4 >_{g=2}$ and $< \tau_7 >_{g=3}$, we obtain with (5.13)

$$\begin{aligned} < \tau_4 >_{g=2} &= \frac{1}{1152}(1 + 56k^2 + 16k^4), \\ < \tau_7 >_{g=3} &= \frac{1}{2073600}(25 + 5508k^2 + 3120k^4 + 192k^6) \end{aligned} \tag{5.17}$$

The intersection numbers for fractional genus, $< \tau_{\frac{5}{2}} >_{g=\frac{3}{2}}$, $< \tau_{\frac{11}{2}} >_{g=\frac{5}{2}}$, ... are expressed as polynomials with odd powers of k and they are given by the residues for the terms of order σ^3, σ^6,... in (5.9).

$$< \tau_{\frac{5}{2}} >_{g=\frac{3}{2}} = \frac{1}{12}(k + k^3) \tag{5.18}$$

In [19], there is a misprint for this term of order $\sigma^{3/2}$, which had been evaluated from Virasoro equations. The above results agree with the Virasoro equations, which will be discussed below. In general, the intersection numbers with one marked point $< \tau_{3g-2} >_g$ are easily computed to all orders by using the formulae (5.12) and (5.13).

We have used the condition corresponding to $p = 2$ and one point, $s = 1$, $(p+1)(2g-1) = 2n+1$ for $< \tau_n >_g$. This condition $3(2g-1) = 2n+1$ implies $n = 3g-2$ and if n is a half integer, then the genus g is also a half integer. Those half integer g appear for non-orientable surfaces, as discussed earlier with the random surfaces generated by antisymmetric matrices [12] in section 5, and it corresponds to the topology of non orientable surfaces such as the projective plane ($g = \frac{1}{2}$), the Klein bottle ($g = 1$), the cross-capped torus ($g = \frac{3}{2}$), etc. [41].

The string equation for the Kontsevich-Penner model has been derived in [19, 22]; it reads

$$\begin{aligned}\frac{\partial F}{\partial t_0} &= \sum_{n=0,1,2,...} (n+\frac{1}{2}) t_{n+1} \frac{\partial F}{\partial t_n} + \sum_{n=\frac{1}{2},\frac{3}{2},...} (n+\frac{1}{2}) t_{n+1} \frac{\partial F}{\partial t_n} \\ &+ \frac{1}{4} t_0^2 - \frac{k}{2} t_{\frac{1}{2}}\end{aligned} \tag{5.19}$$

The free energy F is divided into close and open parts, F^c and F^o. Then, we have

$$\begin{aligned}\frac{\partial F^c}{\partial t_0} &= \frac{1}{4} t_0^2 + \sum_{n=0,1,2,..} (n+\frac{1}{2}) t_{n+1} \frac{\partial F}{\partial t_n}, \\ \frac{\partial F^o}{\partial t_0} &= -\frac{k}{2} t_{\frac{1}{2}} + \sum_{n=\frac{1}{2},\frac{3}{2},...} (n+\frac{1}{2}) t_{n+1} \frac{\partial F}{\partial t_n}\end{aligned} \tag{5.20}$$

The Virasoro equations for open intersection theory for genus zero has been discussed in [16, 17]. The open intersection numbers are defined analogously to the closed case as

$$< \tau_{n_1} \tau_{n_2} \cdots \tau_{n_s} \hat{\sigma}^{\hat{k}} >_g^o = \int_{\overline{M}_{g,\hat{k},s}} \psi_1^{n_1} \psi_2^{n_2} \cdots \psi_s^{n_s} \tag{5.21}$$

The string equation becomes for the open free energy is

$$\frac{\partial F^o}{\partial \hat{t}_0} = \sum_{i=0}^{\infty} \hat{t}_{i+1} \frac{\partial F^o}{\partial \hat{t}_i} + \hat{s} \tag{5.22}$$

which is consistent with (5.20) ($\hat{s}$ is proportional to k, and the difference is due to a different normalization of $\hat{t}_n$). The string equation implies

$$< \tau_0 \prod \tau_{n_i} \hat{\sigma}^{\hat{k}} >_g^o = \sum_j < \tau_{n_j - 1} \prod_{i \neq j} \tau_{n_i} \hat{\sigma}^{\hat{k}} >_g^o \tag{5.23}$$

We will consider the case of two marked points and derive this string equation in the next section.

open p-th spin cuves

The open intersection numbers for the p-th spin curves with boundaries are also given by the addition of a logarithmic potential to (3.3)

$$U(\sigma) = \frac{1}{\sigma}\oint \frac{du}{2\pi} e^{-\frac{c}{p+1}[(u+\frac{\sigma}{2})^{p+1}-(u-\frac{\sigma}{2})^{p+1}]+k\log(u+\frac{\sigma}{2})-k\log(u-\frac{\sigma}{2})} \tag{5.24}$$

Expanding the exponent,

$$\begin{aligned}
&U(\sigma) = \frac{1}{\sigma}\oint \frac{du}{2\pi i} e^{-\sigma u^p} \\
&\times\exp[-\frac{p(p-1)}{3!4}\sigma^3 u^{p-2} - \frac{p(p-1)(p-2)(p-3)}{5!4^2}\sigma^5 u^{p-4} - \cdots] \\
&\times[1 + k(\frac{1}{u}\sigma + \frac{1}{12u^3}\sigma^3 + \frac{1}{80u^5}\sigma^5\cdots) + \frac{1}{2}k^2(\frac{1}{u^2}\sigma^2 + \frac{1}{6u^4}\sigma^4 + \cdots) \\
&+\frac{1}{3!}k^3(\frac{1}{u^3}\sigma^3 + \cdots) + \frac{1}{4!}k^4(\frac{1}{u^4}\sigma^4 + \cdots) + O(k^5)]
\end{aligned} \tag{5.25}$$

By choosing an integration path around the cut, with $x = \sigma u^p$, the above equation becomes

$$\begin{aligned}
&U(\sigma) = \frac{1}{p\sigma^{1+\frac{1}{p}}\pi}\int_0^\infty dx x^{\frac{1}{p}-1}e^{-x} \\
&\times\exp[-\frac{p(p-1)}{3!4}\sigma^{2+\frac{2}{p}}x^{1-\frac{2}{p}} - \frac{p(p-1)(p-2)(p-3)}{5!4^2}\sigma^{4+\frac{4}{p}}x^{1-\frac{4}{p}} + \cdots] \\
&[1 + k(\sigma^{1+\frac{1}{p}}x^{-\frac{1}{p}} + \frac{1}{12}\sigma^{3+\frac{3}{p}}x^{-\frac{3}{p}} + \frac{1}{5\cdot 2^4}\sigma^{5+\frac{5}{p}}x^{-\frac{5}{p}} + \cdots) \\
&+\frac{k^2}{2}(\sigma^{2+\frac{2}{p}}x^{-\frac{2}{p}} + \frac{1}{6}\sigma^{4+\frac{4}{p}}x^{-\frac{4}{p}} + \cdots) + \frac{1}{3!}k^3(\sigma^{3+\frac{3}{p}}x^{-\frac{3}{p}} + \cdots) \\
&+\frac{1}{4!}k^4(\sigma^{4+\frac{4}{p}}x^{-\frac{4}{p}} + \cdots) + ...]
\end{aligned} \tag{5.26}$$

The integration over x gives

$$\begin{aligned}
&U(\sigma) = -(\frac{p-1}{24} + \frac{k^2}{2})\frac{1}{\pi}\sigma^{1+\frac{1}{p}}\Gamma(1-\frac{1}{p}) - (\frac{p}{24}k + \frac{1}{12}k^3)\frac{1}{\pi}\sigma^{2+\frac{2}{p}}\Gamma(1-\frac{2}{p}) \\
&-\frac{1}{144}[-\frac{(p-1)(p-3)(1+2p)}{40} + (3p+1)k^2 + 2k^4]\frac{1}{\pi}\sigma^{3+\frac{3}{p}}\Gamma(1-\frac{3}{p}) + ...
\end{aligned} \tag{5.27}$$

This expansionc provides the following open intersection numbers,

$$< \tau_{1,0} >_{g=1} = \frac{p-1+12k^2}{24}$$

$$< \tau_{2,1} >_{g=\frac{3}{2}}= \frac{1}{24}(pk+2k^3) \quad (p\neq 2), \quad < \tau_{\frac{5}{2}} >_{g=\frac{3}{2}}= \frac{1}{12}(k+k^3) \quad (p=2)$$

$$< \tau_{n,j} >_{g=2}= \frac{1}{p(12)^2}[\frac{(p-1)(p-3)(1+2p)}{40} - (3p+1)k^2 - 2k^4]\frac{\Gamma(1-\frac{3}{p})}{\Gamma(1-\frac{1+j}{p})} \tag{5.28}$$

where the condition $(p+1)(2g-1) = pn + j + 1$ determines n and j. For p=2, $g = 2$, we have from the above expression,

$$< \tau_{4,0} >_{g=2}== \frac{1}{(24)^2 2!}[1 + 56k^2 + 16k^4] \tag{5.29}$$

which agrees with the result of (5.17). The higher order open intersection numbers of p-th spin curves and one marked point are easily evaluated from the expansion of (5.26). The intersection numbers are related to $U(\sigma)$ as in [6]

$$U(\sigma) = \frac{1}{\pi}\sum < \tau_{n,j} >_g \Gamma(1-\frac{1+j}{p})p^{g-1}\sigma^{(2g-1)(1+\frac{1}{p})} \tag{5.30}$$

For $p = 2$, (Kontsevich-Penner model), we have

$$U(\sigma) = \frac{1}{2\pi\sigma^{\frac{3}{2}}}e^{-\frac{1}{12}\sigma^3}\lim_{p\to 2}\int_0^\infty dx e^{-x}x^{\frac{1}{p}-1}\left(\frac{1+\frac{1}{2}\sigma^{1+\frac{1}{p}}x^{-\frac{1}{p}}}{1-\frac{1}{2}\sigma^{1+\frac{1}{p}}x^{-\frac{1}{p}}}\right)^k \tag{5.31}$$

Additional computations of open intersection numbers for p-spin curves are listed in an Appendix. In this Appendix, we derive also the string equation for p spin curves in the presence of a logarithmic potential.

open $O(2N)$ model

We now consider the non orientable intersection numbers provided by the $O(2N)$ model. It is natural to investigate the relation between the non-orientable intersection numbers given by the $O(2N)$ model and the open intersection numbers which we have just discussed. The open intersection numbers for the $O(2N)$ case with a logarithmic potential is also interesting since the model deviates from KdV and KP hierarchies.

For the O(2N) case with a logarithmic potential, $U(\sigma)$ for the p-th higher Airy singularity becomes (4.10),

$$U(\sigma) = \frac{1}{N\sigma}\oint\frac{du}{2\pi i}e^{-\frac{c}{p+1}[(u+\frac{\sigma}{4})^{p+1}-(u-\frac{\sigma}{4})^{p+1}]+k\log(\frac{u+\frac{\sigma}{4}}{u-\frac{\sigma}{4}})}(1-\frac{\sigma}{4u}) \tag{5.32}$$

Since it resembles to the unitary case, with the replacement σ by $2s$, the expansion (5.26) can be used.

$$U(s) = \frac{1}{ps^{1+\frac{1}{p}}\pi}\int_0^\infty dx x^{\frac{1}{p}-1}e^{-x}(1-\frac{1}{2}s^{1+\frac{1}{p}}x^{-\frac{1}{p}})$$

$$\times\exp[-\frac{p(p-1)}{3!4}s^{2+\frac{2}{p}}x^{1-\frac{2}{p}}-\frac{p(p-1)(p-2)(p-3)}{5!4^2}s^{4+\frac{4}{p}}x^{1-\frac{4}{p}}+\cdots]$$
$$[1+k(s^{1+\frac{1}{p}}x^{-\frac{1}{p}}+\frac{1}{12}s^{3+\frac{3}{p}}x^{-\frac{3}{p}}+\frac{1}{5\cdot 2^4}s^{5+\frac{5}{p}}x^{-\frac{5}{p}}+\cdots)$$
$$+\frac{k^2}{2}(s^{2+\frac{2}{p}}x^{-\frac{2}{p}}+\frac{1}{6}s^{4+\frac{4}{p}}x^{-\frac{4}{p}}+\cdots)+\frac{1}{3!}k^3(s^{3+\frac{3}{p}}x^{-\frac{3}{p}}+\cdots)$$
$$+\frac{1}{4!}k^4(s^{4+\frac{4}{p}}x^{-\frac{4}{p}}+\cdots)+...] \tag{5.33}$$

The term $(-\frac{1}{2}s^{1+\frac{1}{p}}x^{-\frac{1}{p}})$ gives an additional contribution to the open intersection numbers characterized by a parameter k as discussed in (5.28). This contribution reads

$$\begin{aligned} U(s) &= U_0(s)+\Delta U(s) \\ \Delta U(s) &= \frac{1}{\pi}[\frac{k}{2}s^{1+\frac{1}{p}}\Gamma(1-\frac{1}{p})+\frac{p-1}{48}s^{2+\frac{2}{p}}\Gamma(1-\frac{2}{p}) \\ &+ \frac{1}{72}(k+2k^3)s^{3+\frac{3}{p}}\Gamma(1-\frac{3}{p})+\cdots] \end{aligned} \tag{5.34}$$

where $U_0(s)$ is the same as $U(\sigma)$ in (5.26). Thus the open intersection numbers for the $O(2N)$ case ($O(2N)$ p-th Airy matrix model with a logarithmic potential), together with $U_0(s)$, are given by

$$\begin{aligned} &< \tau_{1,0} >_{g=1}=\frac{p-1+12k+12k^2}{24} \\ &< \tau_{2,1} >_{g=\frac{3}{2}}=\frac{(p-1)+2pk+6k^2+4k^3}{48}, \quad < \tau_{\frac{5}{2}} >_{g=\frac{3}{2}}=\frac{1+4k+6k^2+4k^3}{48} \\ &< \tau_{n,j} >_{g=2}=\frac{1}{p(12)^2}[\frac{(p-1)(p-3)(1+2p)}{40}+2k-(1+3p)k^2+4k^3-2k^4] \\ &\times\frac{\Gamma(1-\frac{3}{p})}{\Gamma(1-\frac{1+j}{p})} \end{aligned} \tag{5.35}$$

6 Multiple marked points and Virasoro equations

string equation

The Virasoro equations have been investigated for the Kontsevich-Penner model [19, 22]. The first Virasoro equation, or string equation, reads (5.19) and (5.23).

$$< \tau_0\prod_{i=1}^{s}\tau_{n_i} >_g=\sum_{j=1}^{s}< \tau_{n_j-1}\prod_{i\neq j}^{s}\tau_{n_i} >_g \tag{6.1}$$

Since the intersection numbers for s-marked points are known explicitly from the integral formula for$U(\sigma_1, ..., \sigma_s)$, it is interesting to derive the above string equation and the other Virasoro equations for the Kontsevich-Penner model from our formulation of the s-point correlation function $U(\sigma_1, ...\sigma_s)$.

2 *marked points* :

The two marked points correlation function $U(\sigma_1, \sigma_2)$ is [6]

$$U(\sigma_1, \sigma_2) = e^{-\frac{1}{12}(\sigma_1^3+\sigma_2^3)} \oint \frac{du_1 du_2}{(2\pi i)^2} \exp\{-\sigma_1 u_1^2 - \sigma_2 u_2^2 + k\log(\frac{u_1 + \frac{1}{2}\sigma_1}{u_1 - \frac{1}{2}\sigma_1})$$
$$+k\log(\frac{u_2 + \frac{1}{2}\sigma_2}{u_2 - \frac{1}{2}\sigma_2})\} \frac{1}{(u_1 - u_2 + \frac{1}{2}(\sigma_1 + \sigma_2))(u_2 - u_1 + \frac{1}{2}(\sigma_1 + \sigma_2))}. \quad (6.2)$$

Writing the denominators as principal parts integrals as in [45]

$$\frac{1}{u_1 - u_2 + \frac{1}{2}(\sigma_1 + \sigma_2)} = -i \int_0^\infty d\alpha e^{i\alpha(u_1 - u_2 + \frac{1}{2}(\sigma_1+\sigma_2))} \quad (6.3)$$

we obtain

$$U(\sigma_1, \sigma_2) = e^{-\frac{1}{12}(\sigma_1^3+\sigma_2^3)} \oint \frac{du_1 du_2}{(2\pi i)^2} \int_0^\infty d\alpha d\beta \exp\{-\sigma_1 u_1^2 - \sigma_2 u_2^2$$
$$+k\log(\frac{u_1 + \frac{1}{2}\sigma_1}{u_1 - \frac{1}{2}\sigma_1}) + k\log(\frac{u_2 + \frac{1}{2}\sigma_2}{u_2 - \frac{1}{2}\sigma_2}) + i(\alpha - \beta)(u_1 - u_2)$$
$$+i(\alpha + \beta)\frac{1}{2}(\sigma_1 + \sigma_2)\} \quad (6.4)$$

Replacing $u_i \to \frac{1}{\sqrt{\sigma_i}}$ and $\alpha \to \sqrt{\sigma_1\sigma_2}\alpha$ and $\beta \to \sqrt{\sigma_1\sigma_2}\beta$, we obtain

$$U(\sigma_1, \sigma_2) = \sqrt{\sigma_1\sigma_2} e^{-\frac{1}{12}(\sigma_1^3+\sigma_2^3)} \oint \frac{du_1 du_2}{(2\pi i)^2} \int_0^\infty d\alpha d\beta \exp\{-u_1^2 - u_2^2$$
$$+ \quad i(\alpha - \beta)(\sqrt{\sigma_2}u_1 - \sqrt{\sigma_1}u_2) + i\frac{1}{2}(\alpha + \beta)(\sqrt{\sigma_1\sigma_2}(\sigma_1 + \sigma_2)$$
$$+ \quad k\log(\frac{u_1 + \frac{1}{2}\sigma_1^{3/2}}{u_1 - \frac{1}{2}\sigma_1^{3/2}}) + k\log(\frac{u_2 + \frac{1}{2}\sigma_2^{3/2}}{u_2 - \frac{1}{2}\sigma_2^{3/2}})\} \quad (6.5)$$

Keeping the term of order $\sqrt{\sigma_2}$ in the pre factor and putting the σ_2 to 0 in the exponent, we obtain

$$U(\sigma_1, \sigma_2) = (\sqrt{\sigma_2} \oint \frac{du_2}{2\pi i} e^{-u_2^2} \frac{1}{u_2^2}) e^{-\frac{1}{12}\sigma_1^3} \oint \frac{du_1}{2\pi i} \exp[-\sigma_1 u_1^2 + k\log(\frac{u_1 + \frac{1}{2}\sigma_1}{u_1 - \frac{1}{2}\sigma_1})] \quad (6.6)$$

Using the argument of (5.13) for the integration over u_2, we have for the one marked point open intersection number $< \tau_{n-1} >_g$ for the Kontsevich-Penner model,

$$< \tau_0 \tau_n >_g = < \tau_{n-1} >_g \tag{6.7}$$

which gives the results of the string equation (6.1) ; for instance,

$$\begin{aligned} &< \tau_0 \tau_{\frac{1}{2}} >_{g=\frac{1}{2}} = < \tau_{-\frac{1}{2}} >_{g=\frac{1}{2}} = k \\ &< \tau_0 \tau_2 >_{g=1} = < \tau_1 >_{g=1} = \frac{1 + 12k^2}{24} \end{aligned} \tag{6.8}$$

In Appendix, we show that the string equation holds also for the p spin curves.

3 marked points:

The three point correlation function $U(\sigma_1, \sigma_2, \sigma_3)$ is given by

$$U(\sigma_1, \sigma_2, \sigma_3) = e^{-\frac{1}{12}(\sigma_1^3 + \sigma_2^3 + \sigma_3^3)} \oint \prod_{i=1}^{3} \frac{du_i}{2\pi i} e^{-\sum \sigma_i u_i^2} \prod_{i=1}^{3} (\frac{u_i + \frac{1}{2}\sigma_i}{u_i - \frac{1}{2}\sigma_i})^k$$
$$\frac{1}{[u_1 - u_2 + \frac{1}{2}(\sigma_1 + \sigma_2)][u_2 - u_3 + \frac{1}{2}(\sigma_2 + \sigma_3)][u_3 - u_1 + \frac{1}{2}(\sigma_3 + \sigma_1)]} \tag{6.9}$$

The three denominators are replaced by integrals ofver α, β, γ as in (6.3). Changing variables $u_i \to \frac{1}{\sqrt{\sigma_i}}$, $\alpha \to \sqrt{\sigma_1 \sigma_3}\alpha$, $\beta \to \sqrt{\sigma_2 \sigma_3}\beta$, $\gamma \to \sqrt{\sigma_3 \sigma_1}\gamma$, the above expression becomes

$$U(\sigma_1, \sigma_2, \sigma_3) = e^{-\frac{1}{12}(\sigma_1^3 + \sigma_2^3 + \sigma_3^3)} \sqrt{\sigma_1 \sigma_2 \sigma_3} \oint \prod_{i=1}^{3} \frac{du_i}{2\pi i} e^{-\sum u_i^2} \prod_{i=1}^{3} (\frac{u_i + \frac{1}{2}\sigma_i^{3/2}}{u_i - \frac{1}{2}\sigma_i^{3/2}})^k$$
$$\int_0^\infty d\alpha d\beta d\gamma e^{i\alpha(\sqrt{\sigma_2}u_1 - \sqrt{\sigma_1}u_2 + \frac{\sqrt{\sigma_1\sigma_2}}{2}(\sigma_1 + \sigma_2))} e^{i\beta(\sqrt{\sigma_3}u_2 - \sqrt{\sigma_2}u_3 + \frac{\sqrt{\sigma_2\sigma_3}}{2}(\sigma_2 + \sigma_3))}$$
$$e^{i\gamma(\sqrt{\sigma_1}u_3 - \sqrt{\sigma_3}u_1 + \frac{\sqrt{\sigma_3\sigma_1}}{2}(\sigma_3 + \sigma_1))} \tag{6.10}$$

Keeping $\sqrt{\sigma_2}$, and putting the other σ_2 to zero, this expression becomes

$$\sqrt{\sigma_2} e^{-\frac{1}{12}(\sigma_1^3 + \sigma_3^3)} \oint \frac{du_2}{2\pi i} \frac{e^{-u_2^2}}{u_2^2} \oint \frac{du_1 du_3}{(2\pi i)^2} \frac{e^{-u_1^2 - u_3^2}}{\sqrt{\sigma_1}u_3 - \sqrt{\sigma_3}u_1 + \frac{1}{2}\sqrt{\sigma_1\sigma_3}(\sigma_1 + \sigma_3)}$$
$$(\frac{u_1 + \sigma_1^{3/2}}{u_1 - \sigma_1^{3/2}})^k (\frac{u_3 + \sigma_3^{3/2}}{u_3 - \sigma_3^{3/2}})^k$$
$$= \sqrt{\sigma_2} e^{-\frac{1}{12}(\sigma_1^3 + \sigma_3^3)} \oint \frac{du_2}{2\pi i} \frac{e^{-u_2^2}}{u_2^2} \oint \frac{du_1 du_3}{(2\pi i)^2} \frac{e^{-\sigma_1 u_1^2 - \sigma_3 u_3^2}}{u_3 - u_1 + \frac{1}{2}(\sigma_1 + \sigma_3)}$$

$$
\frac{u_1 - u_3 + \frac{1}{2}(\sigma_1+\sigma_3)}{u_1 - u_3 + \frac{1}{2}(\sigma_1+\sigma_3)}(\frac{u_1+\sigma_1}{u_1-\sigma_1})^k(\frac{u_3+\sigma_3}{u_3-\sigma_3})^k
$$
$$
= \sqrt{\sigma_2}(\sigma_1+\sigma_3)U(\sigma_1,\sigma_3) \tag{6.11}
$$

Thus we find a string equation for three marked points of the Kontsevich-Penner model,

$$
< \tau_0 \tau_{n_1}\tau_{n_2} >_g = < \tau_{n_1-1}\tau_{n_2} >_g + < \tau_{n_1}\tau_{n_2-1} >_g . \tag{6.12}
$$

Repeating the same procedure, we have a string equation of s-marked points for the Kontsevich-Penner model,

$$
< \tau_0 \prod_{i=1}^{s} \tau_{n_i} >_g = \sum_{j=1}^{s} < \tau_{n_j-1} \prod_{i\neq j}^{s} \tau_{n_i} >_g \tag{6.13}
$$

W-constraints equation

We consider next the terms of order σ_2 in the expression *2 marked points* $U(\sigma_1,\sigma_2)$. The term σ_2 corresponds to $t_{\frac{1}{2}} \sim \mathrm{tr}\frac{1}{\Lambda}$. Such fractional indices correspond to W constraints [19], which appear in the p-th higher Airy matrix model. Such a fractional index appears also in the non-orientable Lie algebra $O(2N)$ as we have seen. Therefore, the terms $t_{n+\frac{1}{2}}$ are characteristics of open intersection numbers. Keeping the order σ_2 terms in (6.5), and putting the other σ_2 to zero,

$$
\begin{aligned}
U(\sigma_1,\sigma_2) &= \sigma_2\sqrt{\sigma_1}e^{-\frac{1}{12}\sigma_1^3}\int_0^\infty d\alpha d\beta \oint \frac{du_1 du_2}{(2\pi i)^2} e^{-u_1^2-u_2^2+k\log(\frac{u_1+\frac{1}{2}\sigma_1^{3/2}}{u_1-\frac{1}{2}\sigma_1^{3/2}})} \\
&\quad e^{-i(\alpha+\beta)\sqrt{\sigma_1}u_2}(\alpha+\beta)(iu_1) \\
&= (2\sigma_2\oint\frac{du_2}{2\pi i}\frac{e^{-u_2^2}}{u_2^3})\frac{1}{\sigma_1}e^{-\frac{1}{12}\sigma_1^3}\oint\frac{du_1}{2\pi i}u_1 e^{-u_1^2} \\
&\quad (1+\frac{k\sigma_1^{\frac{3}{2}}}{u_1}+\frac{k^2\sigma_1^3}{2u_1^2}+\frac{(k+2k^3)\sigma_1^{\frac{9}{2}}}{u_1^3}+\cdots) \\
&= 2\sigma_2\sigma_1^{\frac{1}{2}}R_3R_0k + \sigma_2\sigma_1^2k^2R_3R_1 + 2\sigma_2\sigma_1^{\frac{7}{2}}R_3(-kR_0+(k+2k^3)R_2)+\cdots
\end{aligned} \tag{6.14}
$$

where we use the following contour integrals,

$$
\begin{aligned}
R_{2n} &= \oint\frac{du}{2\pi i}\frac{1}{u^{2n}}e^{-u^2} = \Gamma(\frac{1}{2}-n) \\
R_{2n+1} &= \oint\frac{du}{2\pi i}\frac{1}{u^{2n+1}}e^{-u^2} = \lim_{p\to 2}\oint\frac{du}{2\pi i}\frac{1}{u^{2n+1}}e^{-u^p} = \lim_{p\to 2}\frac{2}{p}\Gamma(1-\frac{2n}{p}) \\
R_2 &= -2R_0 = -2\Gamma(\frac{1}{2})
\end{aligned} \tag{6.15}
$$

From this expression, we have

$$< \tau_{\frac{1}{2}}\tau_0 >_{g=\frac{1}{2}}= k, \quad < \tau_{\frac{1}{2}}\tau_{\frac{3}{2}} >_{g=1}= k^2, \quad < \tau_{\frac{1}{2}}\tau_3 >_{g=\frac{3}{2}}= \frac{1}{6}(3k+4k^3) \quad (6.16)$$

dilaton equation

Next, we consider the dilaton equation, which involve $\sigma_{\frac{3}{2}}$, i.e. τ_1 in the intersection numbers. The equation $L_0 Z = 0$ is a dilaton equation, which is derived by considering the terms of order $\mathrm{tr}\frac{1}{\Lambda^{\frac{3}{2}}}$. For the dilaton equation, we have

$$< \tau_1 \prod_{i=1}^{s} \tau_{n_i} >_g= (2g-2+s) < \prod_{i=1}^{s} \tau_{n_i} >_g \quad (6.17)$$

From the three point function $U(\sigma_1,\sigma_2,\sigma_3)$, we obtain

$$< \tau_0\tau_{\frac{1}{2}}\tau_1 >_{g=\frac{1}{2}}=< \tau_0\tau_{\frac{1}{2}} >_{g=\frac{1}{2}}= k, \quad < \tau_0^3\tau_1 >_{g=0}=< \tau_0^3 >_{g=0}= 1, ... \quad (6.18)$$

which satisfies the dilaton equation of (6.17).

For $\sigma_2^{\frac{3}{2}}$ in $U(\sigma_1,\sigma_2,...,\sigma_s)$, scaling $u_i \to (\frac{x_i}{\sigma_i})^{\frac{1}{2}}$, it is obvious that the logarithmic term for u_2 can be neglected, since it gives higher orders. Therefore, the dilaton equation does not show the effect of the logarithmic term and the equation is same as the dilaton equation without the logarithmic potential (there is no k in the equation) as (6.17), and thus we find that (6.17) holds.

In Appendix, we discuss the string equation, the divisor equation and the dilaton equation for p spin curves in the presence of the logarithmic potential.

Virasoro equations for Kontsevich-Penner model

The Virasoro equations are expressed through operators L_n, which act on the partition function $Z = e^F$,

$$L_n Z = L_n e^F = 0 \quad (6.19)$$

with $n = -1, 0, 1, ...$; for $n = -1$ it gives the string equation and for $n = 0$ the dilaton equation. The free energy is a generating function of the intersection numbers with a variables $t_n = \mathrm{tr}\frac{1}{\Lambda^{n+\frac{1}{2}}}$. For the Kontsevich-Penner model, since there is a logarithmic potential, we need to consider also half-integer values for n in $t_{\frac{n}{2}}$. We recall previous calculations [19] and summarize here a comparison with the calculation based$U(\sigma_1,...,\sigma_s)$. The first Virasoro equation for the order $\mathrm{tr}\frac{1}{\Lambda^{\frac{1}{2}}}$, which is the string equation, is given by

$$\frac{\partial F}{\partial t_0} = \frac{1}{4}t_0^2 - \frac{k}{2}t_{\frac{1}{2}} + \sum_{n=0,\frac{1}{2},1,...} (n+\frac{1}{2})t_{n+1}\frac{\partial F}{\partial t_n} \quad (6.20)$$

The free energy is a generating function of the intersection numbers as

$$F=\sum_{d_n}<\prod_n \tau_n^{d_n}>\prod_n \frac{\hat{t}_n^{d_n}}{d_n!} \tag{6.21}$$

where

$$\hat{t}_n=\mathrm{tr}\frac{1}{(2^{\frac{2}{3}}\Lambda)^{\mathrm{n}+\frac{1}{2}}} \tag{6.22}$$

We have the relation

$$t_n=(2^{\frac{2}{3}})^{n+\frac{1}{2}}\hat{t}_n \tag{6.23}$$

In [19], the Virasoro equations up to third order (i.e. up to the dilaton equation) are obtained as

$$\begin{aligned}
&(-\frac{\partial}{\partial t_0}+\frac{1}{4}J^{(2)}_{-2}-\frac{k}{2}t_{\frac{1}{2}})g=0 \quad (\textit{string equation})\\
&(-2\frac{\partial}{\partial t_{\frac{1}{2}}}-kt_0-\frac{1}{16}t_{\frac{3}{2}}-\frac{k^2}{4}t_{\frac{3}{2}}-\frac{1}{12}J^{(3)}_{-4}+\frac{k}{4}J^{(2)}_{-4}-\frac{1}{2}J^{(2)}_{-1})g=0\\
&(-3\frac{\partial}{\partial t_1}-\frac{1}{16}-\frac{3}{4}k^2+kt_0t_{\frac{1}{2}}-\frac{1}{4}J^{(2)}_0-\frac{1}{4}J^{(2)}_{-3})g=0 \quad (\textit{dilaton equation})
\end{aligned} \tag{6.24}$$

with

$$\begin{aligned}
J^{(1)}_m &= \frac{\partial}{\partial x_m}-mx_{-m}, \quad (m=...,-2,-1,0,1,2,...)\\
J^{(2)}_m &= \sum_{i+j=m}:J^{(1)}_i J^{(1)}_j:\\
&= \sum_{i+j=m}\frac{\partial^2}{\partial x_i\partial x_j}+2\sum_{-i+j=m}ix_i\frac{\partial}{\partial x_j}+\sum_{-i-j=m}(ix_i)(jx_j)\\
J^{(3)}_m &= \sum_{i+j+k=m}:J^{(1)}_i J^{(1)}_j J^{(1)}_k:\\
&= \sum_{i+j+k=m}\frac{\partial^3}{\partial x_i\partial x_j\partial x_k}+3\sum_{-i+j+k=-4}
\end{aligned} \tag{6.25}$$

where $:\cdots:$ means normal ordering, i.e. pulling the differential operator to the right. $x_n=\frac{1}{n}t_{\frac{n-1}{2}}$. The solution of the Virasoro equations, which includes the string equation, gives [19]

$$F = \frac{1}{12}t_0^3+\frac{1}{48}(1+12k^2)t_1+\frac{1}{2}kt_0t_{\frac{1}{2}} \tag{6.26}$$

$$\begin{aligned}
&+\frac{1}{24}t_0^3t_1+(\frac{1}{192}+\frac{1}{16}k^2)t_1^2+\frac{k}{4}t_0t_{\frac{1}{2}}t_1+\frac{k}{24}(t_{\frac{1}{2}})^3+\frac{1}{96}(1+12k^2)t_0t_2\\
&+\frac{k}{4}t_0^2t_{\frac{3}{2}}+\frac{1}{4}k^2t_{\frac{1}{2}}t_{\frac{3}{2}}+\frac{1}{6}(k+k^3)t_{\frac{5}{2}}+\cdots
\end{aligned} \tag{6.27}$$

From this expression, the intersection numbers , which are defined as (6.21) are obtained by changing t_n to $\hat{t}_n$ for small genus,

$$
\begin{aligned}
&< \tau_0^3 >_{g=0}= 1, \quad < \tau_1 >_{g=1}= \frac{1+12k^2}{24}, \quad < \tau_0\tau_{\frac{1}{2}} >_{g=\frac{1}{2}}= k, \\
&< \tau_0^3\tau_1 >_{g=0}= 1, \quad < \tau_1^2 >_{g=1}= \frac{1+12k^2}{24}, \quad < \tau_0\tau_{\frac{1}{2}}\tau_1 >_{g=\frac{1}{2}}= k, \\
&< \tau_{\frac{1}{2}}^3 >_{g=\frac{3}{2}}= k, \quad < \tau_0\tau_2 >_{g=1}= \frac{1+12k^2}{24}, \quad < \tau_0^2\tau_{\frac{3}{2}} >_{g=\frac{1}{2}}= k, \\
&< \tau_{\frac{1}{2}}\tau_{\frac{3}{2}} >_{g=1}= k^2, \quad < \tau_{\frac{5}{2}} >_{g=\frac{3}{2}}= \frac{1}{12}(k+k^3)
\end{aligned}
\tag{6.28}
$$

These intersection numbers satisfy the string equation, the W-constraints, and the dilaton equation. They provide identical values as those calculated from $U(\sigma_1, ..., \sigma_s)$ for the Kontsevich-Penner model, as shown in Appendix.

7 Gromov-Witten invariants of CP^1 model

The Gromov-Witten invariants of CP^1 model has been studied [46, 47, 48]. Recently, the Gromov-Witten invariants are evaluated in more higher orders [49]. We apply the present method to Gromov-Witten invariants of CP^1 model, since it has a similar matrix model representation as Kontsevich type of the external source [48].

The CP^1 matrix model is described as

$$Z = \int D(e^M) e^{\mathrm{tr}(\mathrm{e^M + qe^{-M}}) + \mathrm{trM}\Lambda} \tag{7.1}$$

We use the Gaussian random matrix model with an external source, and by the tuning the external source A, we obtain (7.1) as a generalized Kontsevich model. Therefore, as before, we consider the Fourier transform of the density correlation functions. Particularly, we consider $U(\sigma)$, which is

$$
\begin{aligned}
U(\sigma) &= \frac{1}{\sigma}\oint \frac{du}{2\pi i} e^{e^{u+\frac{1}{2}\sigma} - e^{u-\frac{1}{2}\sigma} + qe^{-(u+\frac{1}{2}\sigma)} - qe^{-(u-\frac{1}{2}\sigma)} + u} \\
&= \frac{1}{\sigma}\oint \frac{du}{2\pi i} e^{(2\sinh\frac{\sigma}{2})(\mathrm{e^u - qe^{-u}}) + \mathrm{u}}
\end{aligned}
\tag{7.2}
$$

By $x = e^u$, we have

$$U(\sigma) = \frac{1}{\sigma}\oint \frac{dx}{2\pi i} e^{(2N\sinh\frac{\sigma}{2N})(x - qx^{-1})} \tag{7.3}$$

where we inserted N to make clear the genus expansion. The residue calculation becomes

$$U(\sigma)\frac{1}{\sigma}\sum_{d=1} \frac{1}{d!(d-1)!}(2N\sinh\frac{\sigma}{2N})^{2d-1}(-q)^d \tag{7.4}$$

In the genus zero, $N \to \infty$, dropping the irrelevant factor of q, we have

$$
\begin{aligned}
U(\sigma) &= \sum_{d=1}^{\infty} \frac{1}{d!(d-1)!} \sigma^{2(d-1)} \\
&= \sum_{d=0}^{\infty} \frac{1}{(d+1)!d!} \sigma^{2d} = \sum_{d=0}^{\infty} < \tau_{2d} >_{g=0} (d+1)\sigma^{2d}
\end{aligned}
\tag{7.5}
$$

with

$$
< \tau_{2d} >_{g=0} = \frac{1}{(d+1)!^2}. \tag{7.6}
$$

For the higher genus g, we expand $(2N\sinh\frac{\sigma}{2N})^{2d-1}$, and pick up the genus g terms from order $\frac{1}{N^{2g}}$ terms.

$$
\frac{1}{\sigma}(2N\sinh\frac{\sigma}{2N})^{2d-1} = \sigma^{2d-2} + \frac{2d-1}{24N^2}\sigma^{2d} + \frac{(2d-1)(10d-7)}{5760N^4}\sigma^{2d+2} + \cdots \tag{7.7}
$$

With the shift of power of σ, $2d + 2n \to 2d$, we obtain Gromov-Witten invariants of genus g as

$$
U(\sigma) = \sum_d \sum_g < \tau_{2d} >_g (d + 1 - g)\sigma^{2d} \tag{7.8}
$$

with

$$
\begin{aligned}
< \tau_{2d}(\omega) >_{g=0} &= \frac{1}{((d+1)!)^2} \\
< \tau_{2d}(\omega) >_{g=1} &= \frac{2d-1}{24(d!)^2} \\
< \tau_{2d}(\omega) >_{g=2} &= \frac{d^2(2d-3)(10d-17)}{2^7 \cdot 3^2 \cdot 5(d!)^2} \\
< \tau_{2d}(\omega) >_{g=3} &= \frac{d^2(d-1)^2(2d-5)(140d^2-784d+1101)}{2^{10} \cdot 3^4 \cdot 5 \cdot 7(d!)^2}
\end{aligned}
\tag{7.9}
$$

These numbers agree with the result of the recent evaluation by Norbury and Scott by a different method up to genus three[49]. It is straight forward to evaluate Gromov-Witten one point invariants in any order of genus from $U(\sigma)$. We have for $g = 4$ as

$$
< \tau_{2d}(\omega) >_{g=4} = \frac{d^2(d-1)^2(d-2)^2}{(d!)^2} \frac{(2d-7)(10d-39)(140d^2-1092d+2143)}{2^{15} \cdot 3^5 \cdot 5^2 \cdot 7} \tag{7.10}
$$

8 Discussions

In this article, we have considered the generalization of the Airy matrix model to a p-th singularity. This provides the intersection numbers of the moduli space of p-spin curves for orientable and non-orientable Riemann surfaces, with Lie algebras of $U(N)$, $O(2N)$, $O(2N{+}1)$ and $Sp(N)$. The Euler characteristics are easily evaluated by taking the $p \to -1$ limit. Our results are consistent with the two categories, orientable and non-orientable surfaces, since we have obtained two type of topological invariants (two different Euler characteristics) for Lie algebras. The expressions agree with the virtual Euler characteristics obtained earlier [42] for non-orientable surfaces. We have obtained explicit expressions to all order in the genus for one marked point in the $p = 3$ and $p = 4$ cases given in terms of Bessel functions.

For the open intersection numbers, which are defined by the insertion of a disk on a closed Riemann surface as a boundary, we have used the Kontsevich-Penner model. We have derived the Virasoro equations, string equation and dilaton equations, for this Kontsevich-Penner model from explicit integral representations. The open intersection numbers are extended to p–spin curves, from a higher Airy matrix model with a logarithmic potentials.

In our previous article [6], the Airy matrix model with a logarithmic potential was derived from the average of two characteristic polynomials in a two matrix model with an external source The eigenvalues of the first matrix M_1 is on an edge of the distribution, and for the other matrix M_2 in the bulk,. After integration over the matrix M_2, a model with a logarithmic potential is obtained for M_1. The coefficient of the logarithmic potential k corresponds to the power of $(\det M_1)^k$. This logarithmic potential provides the boundary for the open intersection theory.

The integral representation of the s point correlation function for a Gaussian matrix model with an external source provides a powerful tool for the evaluation of open/close intersection numbers and Gromov-Witten invariants. It would be interesting to extend the present analysis to more complicated cases , such as the Gromov-Witten theory of CP^{n-1}.

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Appendix: Virasoro equations of open intersection numbers for p spin curves

The open/close intersection numbers of p-spin curves are evaluated from the expressions for the s-point correlation functions $U(\sigma_1, ..., \sigma_s)$. From (5.26), the one marked point intersection numbers $< \tau_{n,j} >_g$ are obtained as

$$
\begin{aligned}
&< \tau_{1,0} >_1 = \frac{p-1+12k^2}{24} \\
&< \tau_{n,j} >_{\frac{3}{2}} = \frac{1}{24}(pk+2k^3)\frac{\Gamma(1-\frac{2}{p})}{\Gamma(1-\frac{1+j}{p})} \\
&< \tau_{n,j} >_2 = \frac{1}{p(12)^2}\{\frac{(p-1)(p-3)(1+2p)}{40} - (1+3p)k^2 - 2k^4\}\frac{\Gamma(1-\frac{3}{p})}{\Gamma(1-\frac{1+j}{p})} \\
&< \tau_{n,j} >_{\frac{5}{2}} = \{\frac{k}{5760}(18-25p+30p^2-5p^3) + \frac{k^3}{144}(p-1)\}\frac{\Gamma(1-\frac{4}{p})}{\Gamma(1-\frac{1+j}{p})} \\
&< \tau_{n,j} >_3 = \frac{1}{p^2}\{\frac{1}{2903040}(p-1)(p-5)(1+2p)(8p^2-13p-13) \\
&\quad + \frac{1}{57600}(-10p^3+85p^2+90p+19)k^2 + \frac{1}{2880}(5p+3)k^4 + \frac{1}{3600}k^6\} \\
&\quad \times \frac{\Gamma(1-\frac{5}{p})}{\Gamma(1-\frac{1+j}{p})}
\end{aligned}
\tag{A.1}
$$

where n and j are constrained by the condition,

$$
(p+1)(2g-1) = pn + j + 1 \tag{A.2}
$$

i.e. he $s = 1$ case of the general condition,

$$
(p+1)(2g-2+s) = p\sum_{k=1}^{s} n_i + \sum_{k=1}^{s} j_k + s \tag{A.3}
$$

For $< \tau_{\frac{5}{2}} >_{g=\frac{3}{2}}$ of $p = 2$ is obtained from $\lim_{p\to 2} < \tau_{2,1} >_{g=\frac{3}{2}}$ since the right hand side of (A.2) is the same, and it becomes $\frac{1}{12}(k+k^3)$.

For the 2 marked points, the intersection numbers of the p spin curves are derived from $U(\sigma_1, \sigma_2)$.

$$
U(\sigma_1, \sigma_2) \quad = \quad \oint \frac{du_1 du_2}{(2\pi i)^2} e^{-\frac{1}{p}\sum_{i=1}^{2}[(u_i+\frac{1}{2}\sigma_i)^{p+1}-(u_i-\frac{1}{2}\sigma_i)^{p+1}]} \prod_{i=1}^{2}(\frac{u_i+\frac{1}{2}\sigma_i}{u_i-\frac{1}{2}\sigma_i})^k
$$

$$\frac{1}{(u_1 - u_2 + \frac{1}{2}(\sigma_1+\sigma_2))(u_2 - u_1 + \frac{1}{2}(\sigma_1+\sigma_2))} \quad \text{(A.4)}$$

string equation

Using $x_i = \sigma_i u_i^2$, and $\alpha \to (\sigma_1\sigma_2)^{\frac{1}{p}}\alpha$, $\beta \to (\sigma_1\sigma_2)^{\frac{1}{p}}\beta$, taking the same process as (6.5), we obtain

$$\begin{aligned}
&U(\sigma_1,\sigma_2) = \frac{(\sigma_1\sigma_2)^{\frac{1}{p}}}{p^2\pi^2}\int_0^\infty \cdots \int_0^\infty dx_1 dx_2 d\alpha d\beta (x_1x_2)^{\frac{1}{p}-1} e^{-x_1-x_2} \\
&e^{-\frac{p(p-1)}{24}\sum_i (\sigma_i^{2+\frac{2}{p}} x_i^{1-\frac{2}{p}})+\cdots} \\
&e^{i\alpha((\sigma_2x_1)^{\frac{1}{p}}-(\sigma_1x_2)^{\frac{1}{p}}+\frac{1}{2}(\sigma_1\sigma_2)^{\frac{1}{p}}(\sigma_1+\sigma_2))} e^{i\beta((\sigma_1x_2)^{\frac{1}{p}}-(\sigma_2x_1)^{\frac{1}{p}}+\frac{1}{2}(\sigma_1\sigma_2)^{\frac{1}{p}}(\sigma_1+\sigma_2))} \\
&\prod_{i=1}^{2}\left(\frac{x_i^{\frac{1}{p}} + \frac{1}{2}\sigma_i^{1+\frac{1}{p}}}{x_i^{\frac{1}{p}} - \frac{1}{2}\sigma_i^{1+\frac{1}{p}}}\right)^k
\end{aligned} \quad \text{(A.5)}$$

Taking the term of order $\sigma_2^{\frac{1}{p}}$ and neglecting higher order terms in σ_2, we have

$$\begin{aligned}
U(\sigma_1,\sigma_2) &= -\frac{\sigma_2^{\frac{1}{p}}}{\pi}\Gamma(1-\frac{1}{p})\cdot \oint \frac{du_1}{2\pi i} e^{-\frac{1}{p}[(u_1+\frac{1}{2}\sigma_1)^{p+1}-(u_1-\frac{1}{2}\sigma_1)^{p+1}]}\left(\frac{u_1+\frac{1}{2}\sigma_1}{u_1-\frac{1}{2}\sigma_1}\right)^k \\
&= -\frac{\sigma_2^{\frac{1}{p}}}{\pi}\Gamma(1-\frac{1}{p})\cdot \sigma_1 U(\sigma_1)
\end{aligned} \quad \text{(A.6)}$$

This equation is a string equation,

$$< \tau_{0,0}\tau_{n,j} >_g = < \tau_{n-1,j} >_g \quad \text{(A.7)}$$

A string equation for three marked point for p spin curves is an extension of (6.11). It is easily obtained from

$$U(\sigma_1,\sigma_2,\sigma_3) = \sigma_2^{\frac{1}{p}}\Gamma(1-\frac{1}{p})(\sigma_1+\sigma_3)U(\sigma_1,\sigma_3) \quad \text{(A.8)}$$

which is the string equation,

$$< \tau_{0,0}\tau_{n_1,j_1}\tau_{n_2,j_2} >_g = < \tau_{n_1-1,j_1}\tau_{n_2,j_2} >_g + < \tau_{n_1,j_1}\tau_{n_2-1,j_2} >_g \quad \text{(A.9)}$$

W constraint equation

We consider next the W constraint equation. Since there are spin $j = 0,1,...,p-1$ indices for the intersection numbers of p spin curves, we have an equation which involves $\tau_{0,j}$. Taking next $\sigma_2^{\frac{2}{p}}$, and neglecting higher terms in σ_2 in (A.5), we obtain the intersection number with $\tau_{0,1}$.

The two point correlation function $U(\sigma_1,\sigma_2)$ is expressed as [7], by using the following representation.

$$\begin{aligned}&\frac{1}{u_1-u_2+\frac{1}{2}(\sigma_1+\sigma_2)}\frac{1}{u_1-u_2-\frac{1}{2}(\sigma_1+\sigma_2)}\\&=\frac{1}{\sigma_1+\sigma_2}\int_0^\infty d\alpha e^{-\alpha(u_1-u_2)}\sinh(\frac{\alpha}{2}(\sigma_1+\sigma_2))\end{aligned}\tag{A.10}$$

By $\alpha\to(\sigma_1\sigma_2)^{\frac{1}{p}}\alpha$, $u_i=(x_i/\sigma_i)^{\frac{1}{p}}$, $\sinh(\frac{\alpha}{2}(\sigma_1+\sigma_2))\sim\frac{\alpha}{2}(\sigma_1\sigma_2)^{\frac{1}{2}}(\sigma_1+\sigma_2)$, we obtain $\sigma_1^{\frac{1}{p}}$ term as

$$\begin{aligned}U(\sigma_1,\sigma_2)&=\frac{1}{p^2}\frac{(\sigma_1\sigma_2)^{\frac{1}{p}}}{2}\int dx_1dx_2\frac{1}{(\sigma_2x_1)^{\frac{2}{p}}}(x_1x_2)^{\frac{1}{p}-1}e^{-x_1-x_2+\cdots}\\&=\sigma_1^{\frac{1}{p}}\sigma_2U(\sigma_2)\end{aligned}\tag{A.11}$$

which is a string equation. For $\sigma_1^{\frac{2}{p}}$ term, we expand $e^{-(\sigma_1x_2)^{\frac{1}{p}}\alpha}=1+(\sigma_1x_2)^{\frac{1}{p}}\alpha$.

$$U(\sigma_1,\sigma_2)=\frac{1}{p}\sigma_1^{\frac{1}{p}}\Gamma(1-\frac{2}{p})\sigma_2^{-\frac{2}{p}}\int_0^\infty dx_2x_2^{\frac{2}{p}-1}e^{-x_2-\frac{p(p-1)}{24}\sigma_2^{2+\frac{2}{p}}x_2^{1-\frac{2}{p}}+\cdots}(\frac{x_2^{\frac{1}{p}}+\frac{1}{2}\sigma_2^{1+\frac{1}{p}}}{x_2^{\frac{1}{p}}-\frac{1}{2}\sigma_2^{1+\frac{1}{p}}})^k\tag{A.12}$$

From above integral, we obtain $\sigma_1^{\frac{2}{p}}\sigma_2^{4+\frac{2}{p}}$ term as

$$U(\sigma_1,\sigma_2)\sim t_{0,1}t_{4,1}[\Gamma(1-\frac{2}{p})]^2\{(1-\frac{2}{p})\frac{1}{2}\frac{p(p-1)^2}{(24)^2}-\frac{(p-1)(p-2)(p-3)}{5!4^2}\}\tag{A.13}$$

Separating a factor $(1-\frac{2}{p})$, we obtain in the case $k=0$,

$$<\tau_{0,1}\tau_{4,1}>_g=<\tau_{3,2}>_g+\frac{1}{2p}<\tau_{1,0}>_g^2\tag{A.14}$$

for general p. Similarly we obtain for $g=3$, when $k=0$,

$$<\tau_{0,1}\tau_{6,3}>_g=<\tau_{5,4}>_{g=3}+\frac{1}{p}<\tau_{1,0}>_{g=1}<\tau_{3,2}>_{g=2}\tag{A.15}$$

dilaton equation

The dilaton equation for p-spin curves is

$$<\tau_{1,0}\prod_{k=1}^s\tau_{n_k,j_k}>_g=(2g-2+s)<\prod_{k=1}^s\tau_{n_k\cdot j_k}>_g\tag{A.16}$$

We consider $s = 1$, two point correlation function $U(\sigma_1, \sigma_2)$. By the shift $\alpha \to (\sigma_1\sigma_2)^{\frac{1}{p}}$, $x_i = \sigma_i u_i^p$, we have

$$\begin{aligned} U(\sigma_1,\sigma_2) &= \frac{1}{\sigma_1+\sigma_2}\int_0^\infty d\alpha \sinh(\frac{\alpha}{2}(\sigma_1\sigma_2)^{\frac{1}{2}}(\sigma_1+\sigma_2))e^{-[(\sigma_2 x_1)^{\frac{1}{p}}-(\sigma_1 x_2)^{\frac{1}{p}}]+O(\sigma^{2+\frac{2}{p}})} \\ & \prod(\frac{x_i+\frac{1}{2}\sigma_i^{\frac{3}{2}}}{x_i-\frac{1}{2}\sigma_i^{\frac{3}{2}}})^k \end{aligned} \tag{A.17}$$

For simplicity, we evaluate the $p = 2$ case. The term of order $\sigma_1^{\frac{3}{2}}$ comes from

$$\frac{1}{\sigma_1+\sigma_2}\sinh(\frac{\alpha}{2}(\sigma_1\sigma_2)(\sigma_1+\sigma_2))e^{\alpha(\sigma_1 x_2)^{\frac{1}{2}}} \sim \frac{1}{4}\alpha^3\sigma_1^{\frac{3}{2}}\sigma_2^{\frac{1}{2}}x_2 + \frac{\alpha^3}{48}\sigma_1^{\frac{3}{2}}\sigma_2^{\frac{7}{2}} \tag{A.18}$$

By the integrations of α and x_1, we obtain the order of $\sigma_1^{\frac{3}{2}}$

$$U(\sigma_1,\sigma_2) \sim 2\sigma_1^{\frac{3}{2}}\Gamma(1-\frac{1}{2})[\sigma_2^{-\frac{3}{2}}\int dx_2 x_2^{\frac{1}{2}}e^{-x_2+\cdots} + \frac{1}{12}\sigma_2^{\frac{3}{2}}\int dx_2 x_2^{-\frac{1}{2}}e^{-x_2+\cdots}] \tag{A.19}$$

Noting the integral by parts for the first term, we have

$$U(\sigma_1,\sigma_2) = 2\sigma_1^{\frac{3}{2}}\Gamma(1-\frac{1}{2})(1+\frac{1}{6}\sigma_2^3)U(\sigma_2) \tag{A.20}$$

Since we have

$$U(\sigma_2) = \sum_g \frac{1}{(12)^g g!}(-1)^g \sigma_2^{3g-\frac{3}{2}} \tag{A.21}$$

we resum the two terms of (A.20) as

$$U(\sigma_1,\sigma_2) = 2\sigma_1^{\frac{3}{2}}\Gamma(\frac{1}{2})(\sum_{g=1}^\infty \frac{(-1)^g}{(12)^g g!}\sigma_2^{3(g-\frac{1}{2})} + \sum_{g=1}^\infty \frac{(-1)^{g+1}}{(12)^g g!}\frac{1}{6}\sigma_2^{3(g+\frac{1}{2})}) \tag{A.22}$$

which provides the dilaton equation for $p = 2$,

$$< \tau_{1,0}\tau_{n,j} >_g = (2g-1) < \tau_{n,j} >_g \tag{A.23}$$

We can check , for instance, $g = 1$ case for $p = 2$ as

$$< \tau_{1,0}^2 >_{g=1} = < \tau_{1,0} >_{g=1} = \frac{1}{24}(1+12k^2) \tag{A.24}$$

The above equation may be easily extended to $p > 2$ by the same process.

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